Hardness of Approximation for Metric Clustering

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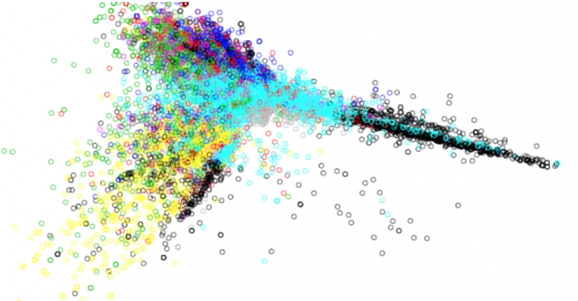
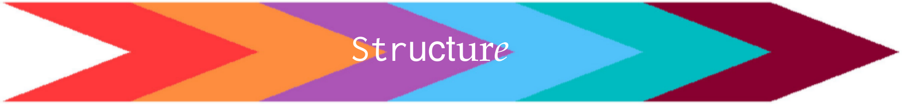
March 5th 2022



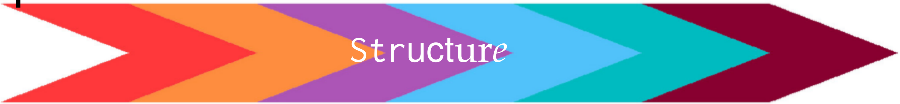
Spectrum of Computational Problems



Spectrum of Computational Problems

Circuit-SAT

CSP



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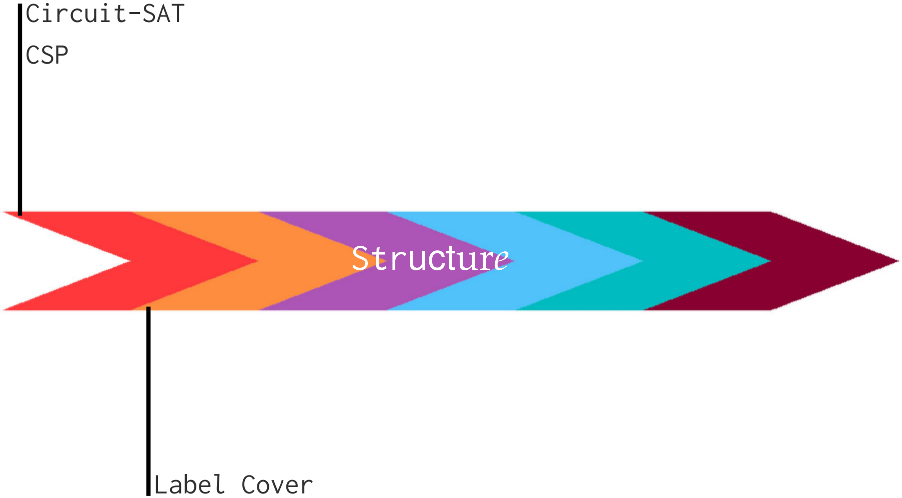

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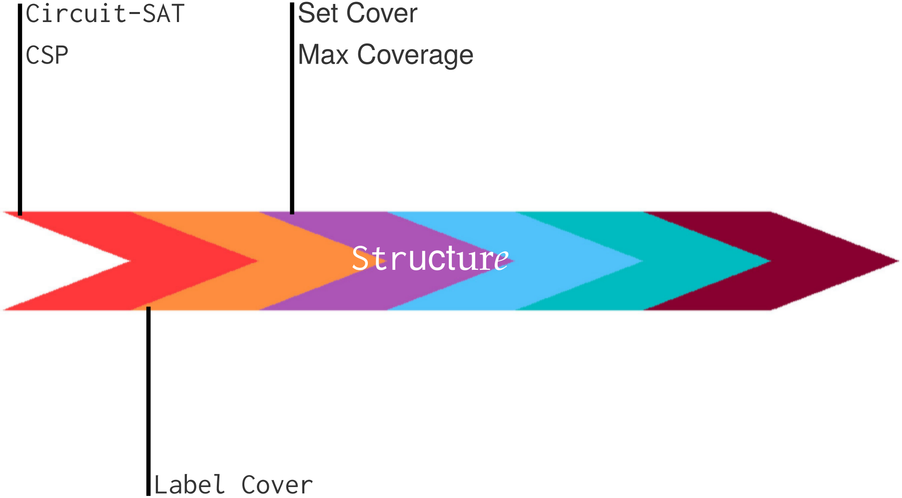

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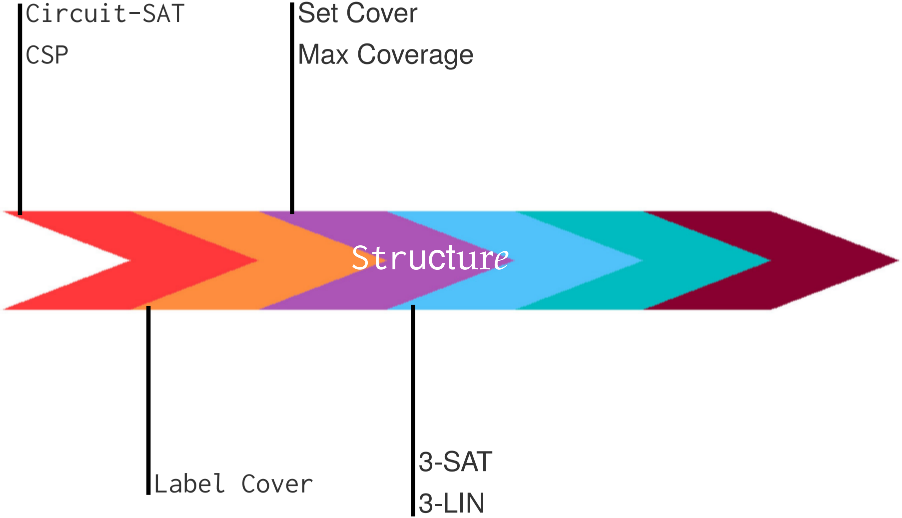

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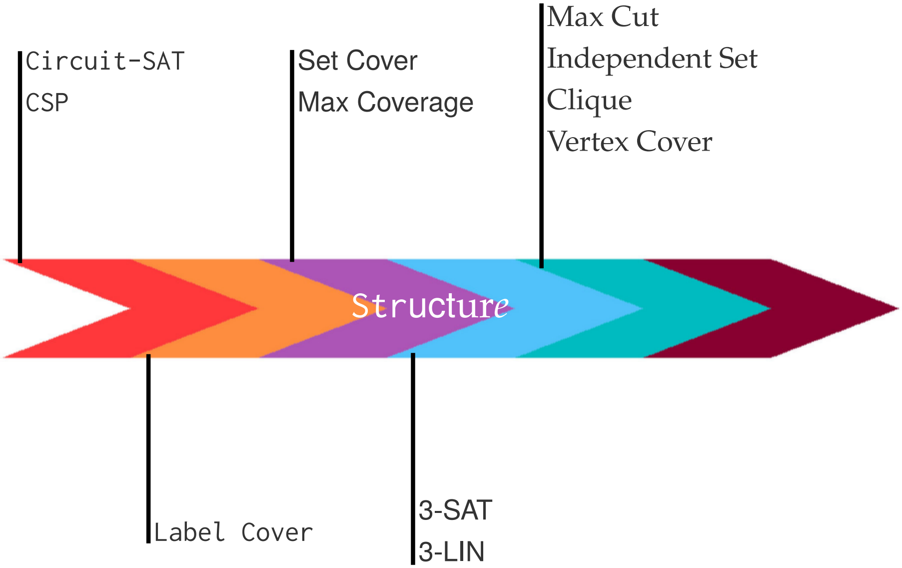

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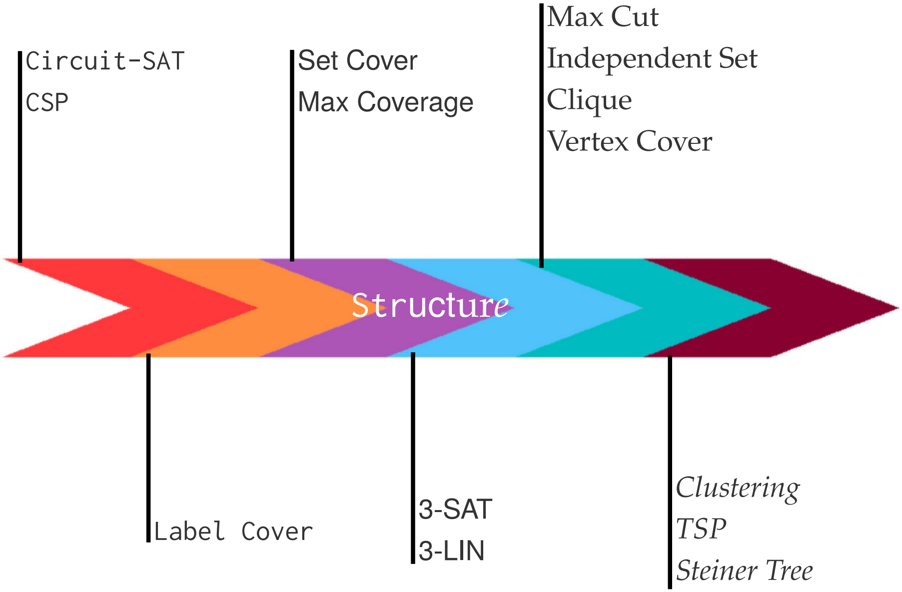

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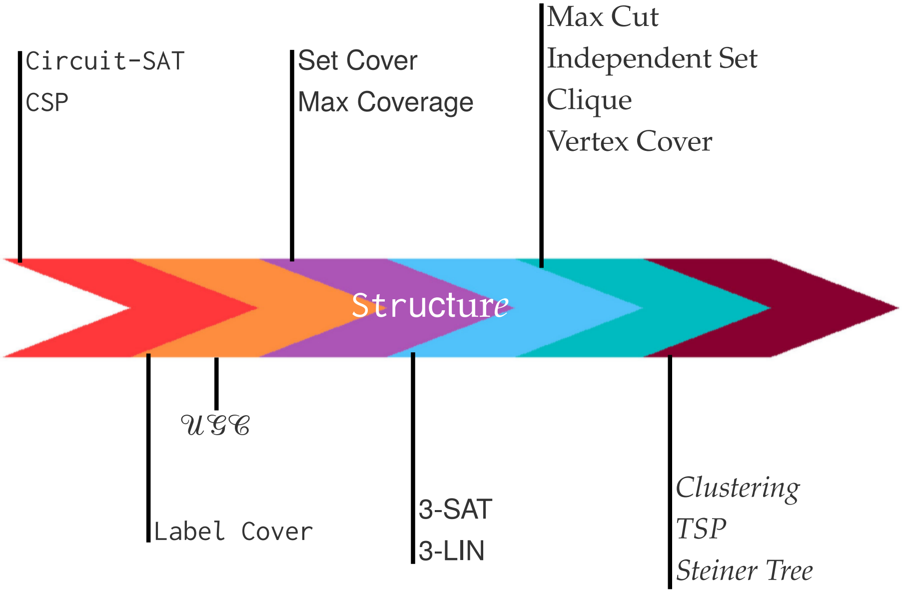

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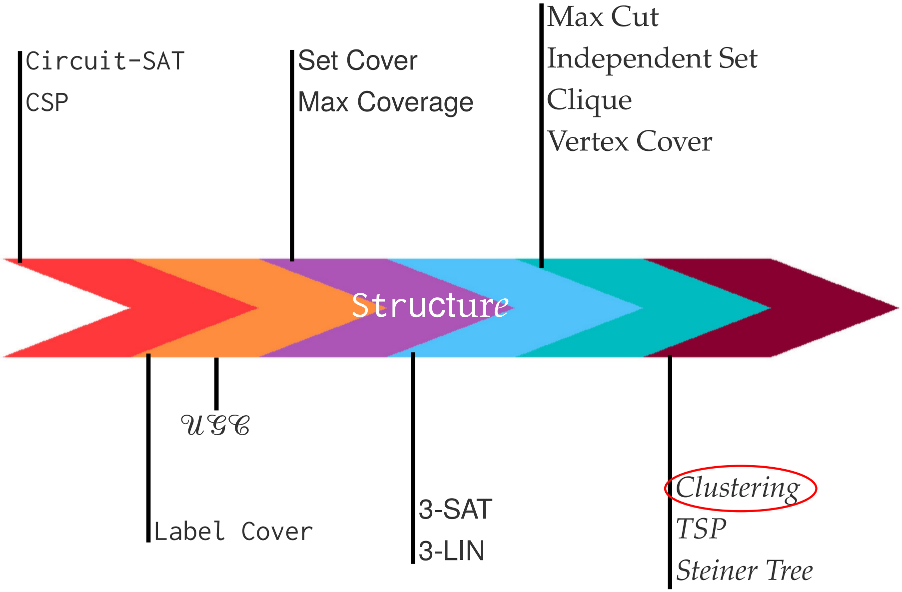

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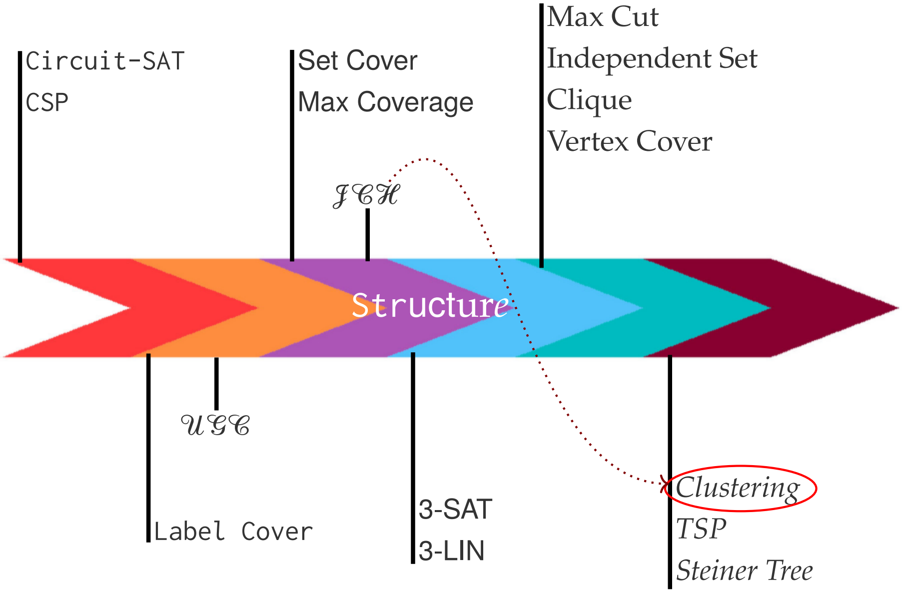

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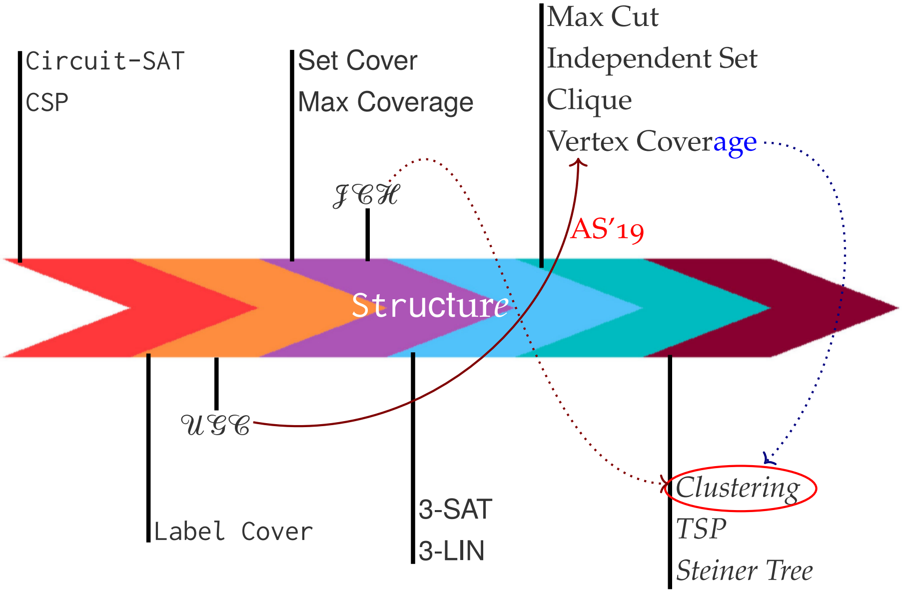


k-center

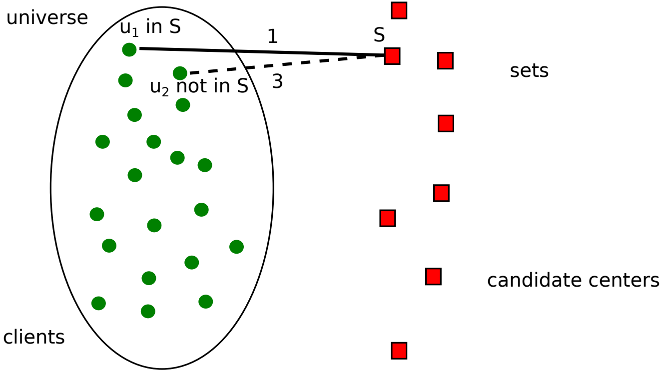
k-center modeling

- Input: $X, S \subseteq \mathbb{R}^d, k \in \mathbb{N}$
- Output: A classification (C, σ):
  - $C \subseteq S$ and $|C| = k$
  - $\sigma : X \to C$
  - (C, σ) minimizes $\max_{x \in X} \|x - \sigma(x)\|_p$

State-of-the-art: General Metrics

- NP-hard [FPT81]
- Poly Time 3-approximation (Gonzalez Algorithm)
- NP-Hard to approximate to $3 - o(1)$ factor! [FPT81]

Proof Overview: General Metrics



Proof Overview: General Metrics

Theorem (Fowler-Paterson-Tanimoto'81)

Given input (X, S, k). It is NP-hard to distinguish:

YES: There exists (C^*, σ^*) such that $\max_{x \in X} \Delta(x, \sigma^*(x)) \leq 1$

NO: For all (C, σ) we have $\max_{x \in X} \Delta(x, \sigma(x)) \geq 3$

State-of-the-art: ℓ_p Metrics

- ℓ_1 and ℓ_∞ metrics
  - Poly Time 3-approximation
  - NP-Hard to approximate to $3 - o(1)$ factor! [FG88]
- Euclidean metric
  - Poly Time 2.74-approximation! [NSS13]
  - NP-Hard to approximate to 2.65 factor [FG88]

Proof Overview: ℓ_p Metrics

Vertex Coverage:

- Input: $G(V, E),\ k$

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$\Downarrow$

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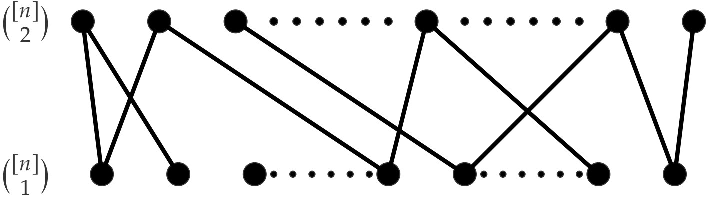
$$\Downarrow$$

Theorem (Fowler-Paterson-Tanimoto'81)

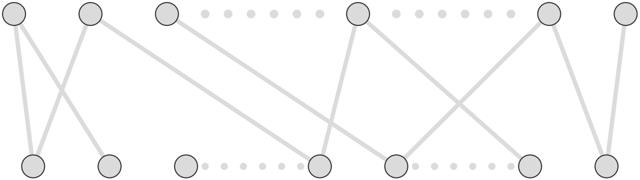
Fix $\varepsilon > 0$. Given input (X, S, k) in $\mathbb{R}^n$. It is NP-hard to distinguish:

YES: There exists (C^*, σ^*) such that $\max_{x \in X} \|x - \sigma^*(x)\|_1 \leq 1$

NO: For all (C, σ) we have $\max_{x \in X} \|x - \sigma(x)\|_1 \geq 3$

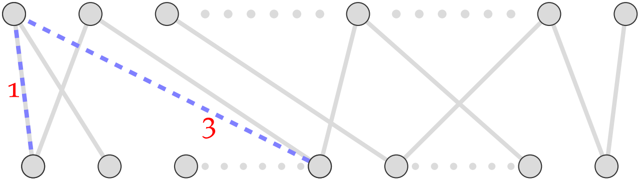

$\binom{[n]}{2}$
$\binom{[n]}{1}$

Points in $\{0,1\}^n$



Graph Embedding

Points in $\{0, 1\}^n$



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k-means & k-median

k-means and k-median modeling

- Input: $X, S \subseteq \mathbb{R}^d$, $k \in \mathbb{N}$
- Output: A classification (C, σ):
  - $C \subseteq S$ and $|C| = k$
  - $\sigma : X \to C$
  - k-means: (C, σ) minimizes $\sum_{x \in X} \|x - \sigma(x)\|_p^2$
  - k-median: (C, σ) minimizes $\sum_{x \in X} \|x - \sigma(x)\|_p$

Our Results (Cohen-Addad–K'19,Cohen-Addad–K–Lee'22)

Discrete Version

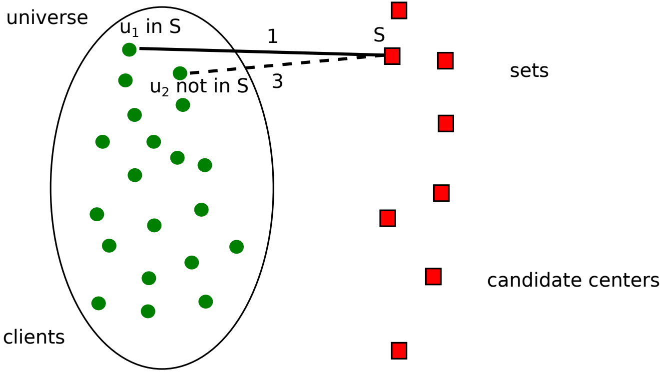
	k-means (JCH)	k-median (JCH)	k-means (UGC)	k-median (UGC)
ℓ_1-metric	3.94	1.73	1.56	1.14
ℓ_2-metric	1.73	1.27	1.17	1.06
ℓ_∞-metric	3.94	1.73	3.94*	1.73*

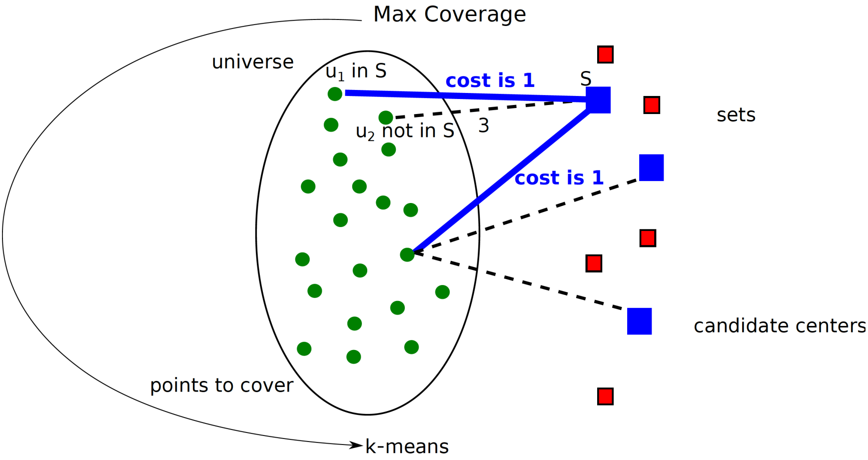
Continuous Version

k-means in ℓ_2-metric $\approx$ 1.36 (JCH), 1.07 (UGC)

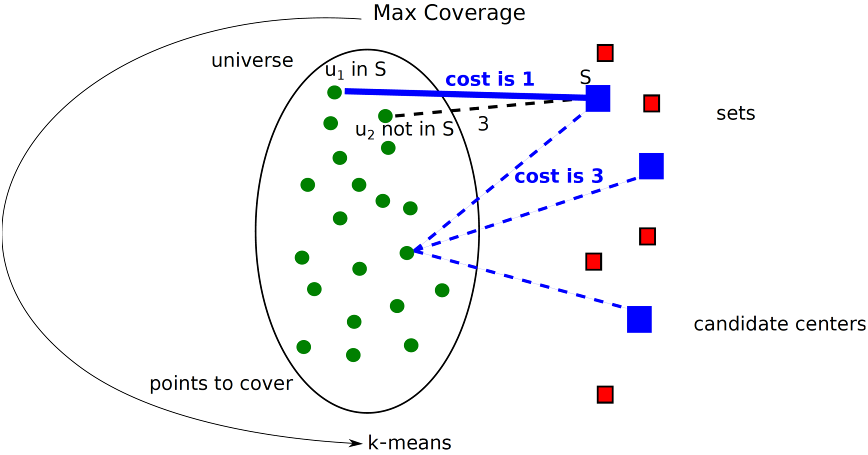
k-median in ℓ_1-metric $\approx$ 1.36 (JCH), 1.07 (UGC)

Proof Overview: General Metrics




Max Coverage
universe
u_1 in S
cost is 1
S
u_2 not in S
3
sets
cost is 1
candidate centers
points to cover
k-means

Proof Overview: General Metrics



Proof Overview: General Metrics

Theorem (Guha-Khuller'99)

Fix $\varepsilon > 0$. Given input (X, S, k). It is NP-hard to distinguish:

YES: There exists (C^*, σ^*) such that $\sum_{x \in X} \Delta(x, \sigma^*(x))^2 \leq |X|$

NO: For all (C, σ) we have $\sum_{x \in X} \Delta(x, \sigma(x))^2 \geq (1 + 8/e - \varepsilon) \cdot |X|$

Johnson Coverage Hypothesis

(α, t)-Johnson Coverage Problem

Given $E \subseteq \binom{[n]}{t}$, and k as input, distinguish between:

Completeness: There exists $\mathscr{C} := \{S_1, \ldots, S_k\} \subseteq \binom{[n]}{t-1}$ such that

$$\forall T \in E,\ \exists S_i \in \mathscr{C},\ S_i \subset T.$$

Soundness: For every $\mathscr{C} := \{S_1, \ldots, S_k\} \subseteq \binom{[n]}{t-1}$ we have

$$\Pr_{T \sim E}[\exists S_i,\ S_i \subset T] \leq \alpha.$$

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Johnson Coverage Hypothesis (Cohen-Addad–K–Lee'22)

$\forall \varepsilon > 0$, $\exists t_\varepsilon \in \mathbb{N}$ such that $(1 - \frac{1}{e} + \varepsilon, t_\varepsilon)$-Johnson Coverage problem is NP-hard.

Theorem (Cohen-Addad–K–Lee'22)

Assuming (α, t)-Johnson coverage problem is NP-hard,

given input $X, \mathcal{S} \subseteq \{0,1\}^{O(\log n)}$, it is NP-hard to distinguish:

Embedding in Hamming metric

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Embedding in Hamming metric

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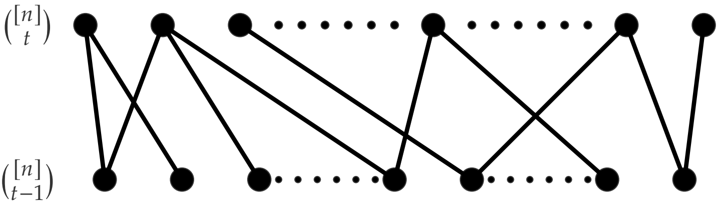
Johnson Graph Embedding



Johnson Graph Embedding

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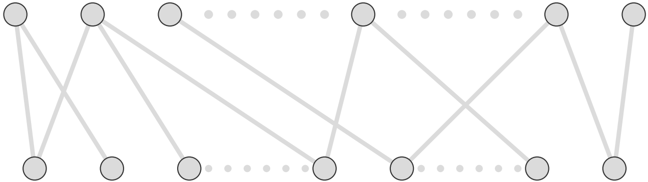

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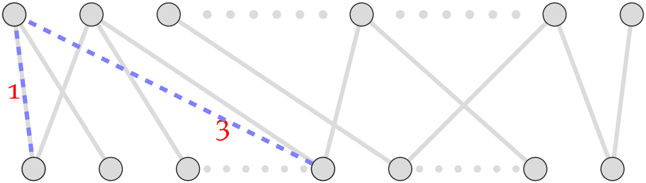
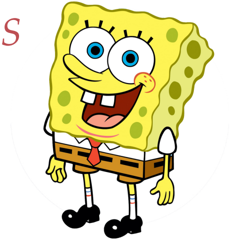


Containment Game

Containment Game

$T \in \binom{[n]}{t}$

$\binom{[n]}{t-1} \ni S$



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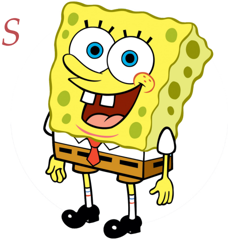

Public Randomness

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Public Randomness

GOAL

Determine if $S \subset T$

Containment Game: Protocols

- Deterministic Protocol:
  - Message length: $O(t \log n)$ bits
  - Completeness: 1, Soundness: 0

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Containment Game: Randomized Protocol

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- Message length: $(t-1) \cdot \log_2 q$
- Soundness: $t \cdot (1 - \Delta(\mathscr{C})) \approx O_t(1/\sqrt{q})$ (for AG codes)

Embedding Transcript into Hamming metric

- Construct $\tau : 2^{[n]} \to \{0,1\}^{q \cdot c \cdot \log n}$

$$\vdots$$

Embedding Transcript into Hamming metric

- Construct $\tau : 2^{[n]} \to \{0,1\}^{q \cdot c \cdot \log n}$
- Fix $i \in [c \cdot \log n]$ and $S \in 2^{[n]}$:

Embedding Transcript into Hamming metric

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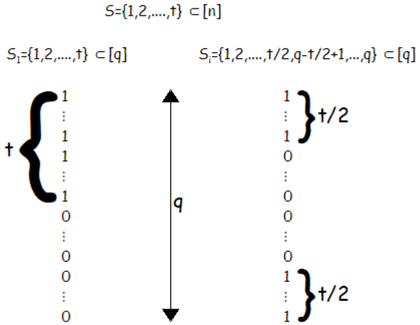

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- $X = \{\tau(T) \mid T \in E\}$
- $\mathcal{S} = \left\{\tau(S) \,\middle|\, S \in \binom{[n]}{t-1}\right\}$

Structural Observations

Suppose $S \subset T$

For every block i, we have $S_i \subset T_i$

$S_i = \{1,2,\ldots,t/2, q-t/2+1,\ldots,q\} \subset [q]$ $\qquad$ $T_i = S_i \cup \{t+1\} \subset [q]$

$$\begin{array}{ccc} 1 & \quad & 1 \\ \vdots & & \vdots \\ 1 & & 1 \\ 0 & & 0 \\ \vdots & & \vdots \\ 0 & & 0 \\ 0 & & 1 \\ 0 & & 0 \\ \vdots & & \vdots \\ 0 & & 0 \\ 1 & & 1 \\ \vdots & & \vdots \\ 1 & & 1 \end{array}$$

$$\boxed{|\tau(T\_i)-\tau(S\_i)|=1}$$

Structural Observations

Suppose $S \not\subset T$

For most blocks i, we have $S_i \not\subset T_i$

$S_i \setminus T_i = \{q\}$	$T_i \setminus S_i = \{t+1,t+2\}$
1	1
⋮	⋮
1	1
0	0
⋮	⋮
0	0
0	1
0	1
0	0
⋮	⋮
0	0
1	1
⋮	⋮
1	1
1	0

$$\boxed{|\tau(T\_i)-\tau(S\_i)| \geq 3}$$

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Our Embedding in Hamming metric

Theorem (Cohen-Addad–K–Lee'22)

Assuming (α, t)-Johnson coverage problem is NP-hard,

given input $X, \mathcal{S} \subseteq \{0,1\}^{O(\log n)}$, it is NP-hard to distinguish:

YES: There exists (C^*, σ^*) such that

$$\sum_{x \in X} \|(x - \sigma^*(x)\|_0^2 \leq n',$$

NO: For all (C, σ) we have

$$\sum_{x \in X} \|(x - \sigma(x)\|_0^2 \geq (1 + 8 \cdot (1 - \alpha)) \cdot n'.$$

Our Results (Cohen-Addad–K'19,Cohen-Addad–K–Lee'22)

Discrete Version

	k-means (JCH)	k-median (JCH)	k-means (UGC)	k-median (UGC)
ℓ_1-metric	3.94	1.73	1.56	1.14
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Decoding Vertex Cover

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  - Recently resolved for $t = 3$
  - Improved SDP gaps for Clustering

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- 3-Hypergraph Vertex Coverage problem is NP-Hard to approximate to a factor of $7/8$

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Open: Is JCH true?

O
PROBLEMS
E
N

State-of-the-art for k-means

Discrete Version

	JCH	UGC	NP≠P
ℓ_1-metric	3.94	1.56	1.38
ℓ_2-metric	1.73	1.17	1.17
ℓ_∞-metric	3.94	3.94	3.94

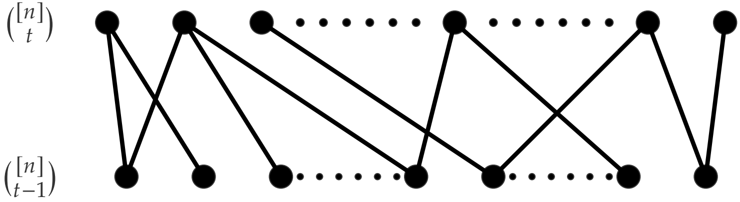
Continuous Version

General metric ≈ 4 (NP≠P)

ℓ_2-metric ≈ 1.36 (JCH), 1.07 (UGC), 1.06 (NP≠P)

ℓ_1-metric ≈ 2.10 (JCH), 1.16 (NP≠P)

ℓ_∞-metric $\approx$???


$\binom{[n]}{t}$
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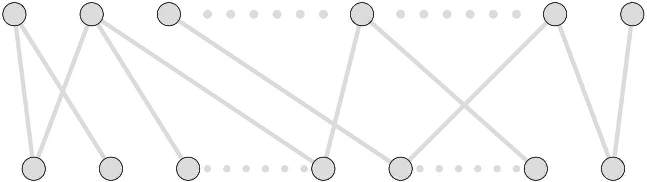
Points in $\{0, 1\}^d$



Inapproximability of Clustering in Euclidean metrics

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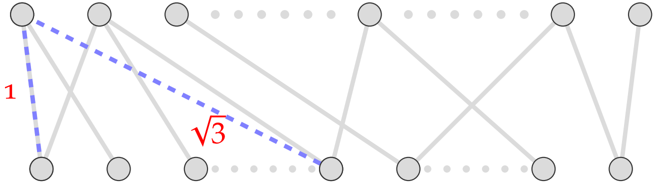

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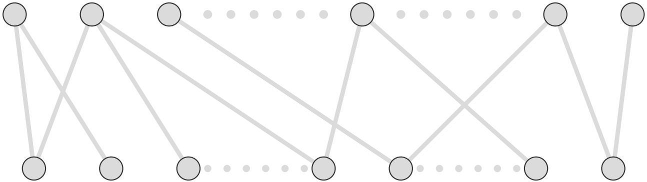

Is there a better embedding of the Johnson Graph into the Euclidean metric?

Tight inapproximability of k-center in Euclidean metrics?

Other Open Problems

Can we prove strong inapproximability results for:

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THANK
YOU!