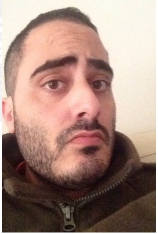
Complexity of Closest Pair via Polar-Pair of Point-Sets

Karthik C. S.
(Weizmann Institute of Science)

Joint work with



Bundit Laekhanukit
(Max-Planck- Institute and Shanghai University of Finance and Economics)

Roee David
(Datorama)

- Closest Pair problem in ℓ_p-metric

Closest Pair

- Closest Pair problem in ℓ_p-metric

  Input: $A \subset \mathbb{R}^d$, $|A| = n$

Closest Pair

- Closest Pair problem in ℓ_p-metric

  Input: $A \subset \mathbb{R}^d$, $|A| = n$

  Output: $a^*, b^* \in A$, $\min_{\substack{a,b \in A \\ a \neq b}} \|a - b\|_p$

Closest Pair

- Closest Pair problem in ℓ_p-metric

  Input: $A \subset \mathbb{R}^d$, $|A| = n$

  Output: $a^*, b^* \in A$, $\min_{\substack{a,b \in A \\ a \neq b}} \|a - b\|_p$

- Trivial algorithm: $O(n^2 d)$

Closest Pair

- Closest Pair problem in ℓ_p-metric

  Input: $A \subset \mathbb{R}^d$, $|A| = n$

  Output: $a^*, b^* \in A$, $\min_{\substack{a,b \in A \\ a \neq b}} \|a - b\|_p$

- Trivial algorithm: $O(n^2 d)$

  Bently-Shamos'76: $2^{O(d)} n \log n$ (for ℓ_2-metric)

Closest Pair

- Closest Pair problem in ℓ_p-metric

  Input: $A \subset \mathbb{R}^d$, $|A| = n$

  Output: $a^*, b^* \in A$, $\min_{\substack{a,b \in A \\ a \neq b}} \|a - b\|_p$

- Trivial algorithm: $O(n^2 d)$

  Bently-Shamos'76: $2^{O(d)} n \log n$ (for ℓ_2-metric)

  Subcubic algorithms when $d = O(n)$: Indyk-Lewenstein-Lipsky-Porat'04, Min-Kao-Zhu'09, Gold-Sharir'17

Closest Pair

- Closest Pair problem in ℓ_p-metric
  Input: $A \subset \mathbb{R}^d$, $|A| = n$
  Output: $a^*, b^* \in A$, $\min_{\substack{a,b \in A \\ a \neq b}} \|a - b\|_p$

- Trivial algorithm: $O(n^2 d)$
  Bently-Shamos'76: $2^{O(d)} n \log n$ (for ℓ_2-metric)
  Subcubic algorithms when $d = O(n)$: Indyk-Lewenstein-Lipsky-Porat'04, Min-Kao-Zhu'09, Gold-Sharir'17

- What happens when $d \approx \log n$?

Bichromatic Closest Pair

- Bichromatic Closest Pair problem in ℓ_p metric

Bichromatic Closest Pair

- Bichromatic Closest Pair problem in ℓ_p metric

  Input: $A, B \subset \mathbb{R}^d$, $|A| = |B| = n$

Bichromatic Closest Pair

- Bichromatic Closest Pair problem in ℓ_p metric

  Input: $A, B \subset \mathbb{R}^d$, $|A| = |B| = n$

  Output: $a^* \in A, b^* \in B, \min_{\substack{a \in A \\ b \in B}} \|a - b\|_p$

Bichromatic Closest Pair

- Bichromatic Closest Pair problem in ℓ_p metric

  Input: $A, B \subset \mathbb{R}^d$, $|A| = |B| = n$

  Output: $a^* \in A, b^* \in B, \min_{\substack{a \in A \\ b \in B}} \|a - b\|_p$

- Trivial algorithm: $O(n^2 d)$

Bichromatic Closest Pair

- Bichromatic Closest Pair problem in ℓ_p metric

  Input: $A, B \subset \mathbb{R}^d$, $|A| = |B| = n$

  Output: $a^* \in A, b^* \in B, \min\limits_{\substack{a \in A \\ b \in B}} \|a - b\|_p$

- Trivial algorithm: $O(n^2 d)$

- Computationally equivalent to determining Minimum Spanning Tree in ℓ_p-metric [Agarwal-Edelsbrunner-Schwarzkopf-Welzl'91, Krznaric-Levcopoulos-Nilsson'99]

Bichromatic Closest Pair

- Bichromatic Closest Pair problem in ℓ_p metric

  Input: $A, B \subset \mathbb{R}^d$, $|A| = |B| = n$

  Output: $a^* \in A, b^* \in B$, $\min_{\substack{a \in A \\ b \in B}} \|a - b\|_p$

- Trivial algorithm: $O(n^2 d)$

- Computationally equivalent to determining Minimum Spanning Tree in ℓ_p-metric [Agarwal-Edelsbrunner-Schwarzkopf-Welzl'91, Krznaric-Levcopoulos-Nilsson'99]

- Assuming SETH, Bichromatic Closest Pair in ℓ_p-metric cannot be solved in subquadratic time when $d = \omega(\log n)$ [Alman-Williams'15]

Our Result

Bichromatic Closest Pair is at least as hard as Closest Pair

Our Result

Bichromatic Closest Pair is at least as hard as Closest Pair

Theorem (Informal)

In every ℓ_p-metric, Bichromatic Closest Pair is computationally equivalent to Closest Pair when $d = \Omega\left(\mathsf{cd}_p\left(K_{n,n}\right)\right)$.

Sphericity of a Graph ($\mathrm{sph}_p(G)$)

Sphericity and Contact Dimension

Sphericity of a Graph ($\mathsf{sph}_p(G)$)

Smallest dimension for which we can realize:

$$\|u - v\|_p \leq 1 \textit{ if } (u, v) \in G \quad \textit{and} \quad \|u - v\|_p > 1 \textit{ otherwise}$$

Sphericity and Contact Dimension

Sphericity of a Graph ($\text{sph}_p(G)$)

Smallest dimension for which we can realize:

$$\|u - v\|_p \leq 1 \text{ if } (u, v) \in G \quad \text{and} \quad \|u - v\|_p > 1 \text{ otherwise}$$

Contact Dimension of a Graph ($\text{cd}_p(G)$)

Sphericity and Contact Dimension

Sphericity of a Graph ($\mathrm{sph}_p(G)$)

Smallest dimension for which we can realize:

$$\|u - v\|_p \leq 1 \textit{ if } (u, v) \in G \quad \textit{and} \quad \|u - v\|_p > 1 \textit{ otherwise}$$

Contact Dimension of a Graph ($\mathrm{cd}_p(G)$)

Smallest dimension for which we can realize:

$$\|u - v\|_p = 1 \textit{ if } (u, v) \in G \quad \textit{and} \quad \|u - v\|_p > 1 \textit{ otherwise}$$

Sphericity and Contact Dimension

Sphericity of a Graph ($\mathrm{sph}_p(G)$)

Smallest dimension for which we can realize:

$$\|u - v\|_p \leq 1 \text{ if } (u, v) \in G \quad \text{and} \quad \|u - v\|_p > 1 \text{ otherwise}$$

Contact Dimension of a Graph ($\mathrm{cd}_p(G)$)

Smallest dimension for which we can realize:

$$\|u - v\|_p = 1 \text{ if } (u, v) \in G \quad \text{and} \quad \|u - v\|_p > 1 \text{ otherwise}$$

$$\mathrm{sph}_p(G) \leq \mathrm{cd}_p(G)$$

Our Result

Theorem (Informal)

In every ℓ_p-metric, Bichromatic Closest Pair is computationally equivalent to Closest Pair when $d = \Omega\left(\mathsf{cd}_p\left(K_{n,n}\right)\right)$.

Proof Sketch

- (n, d, A, B) be instance of Bichromatic Closest Pair in ℓ_p-metric

Proof Sketch

- (n, d, A, B) be instance of Bichromatic Closest Pair in ℓ_p-metric
- $d^* = \mathsf{cd}_p(K_{n,n})$

Proof Sketch

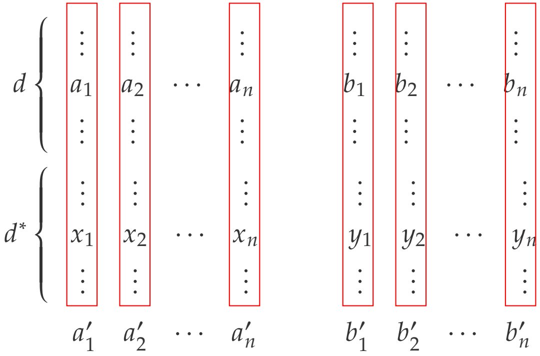
- (n, d, A, B) be instance of Bichromatic Closest Pair in ℓ_p-metric
- $d^* = \mathsf{cd}_p(K_{n,n})$
- $X, Y \subset \mathbb{R}^{d^*}, |X| = |Y| = n \longrightarrow$ "polar-pair of point-sets"

Proof Sketch

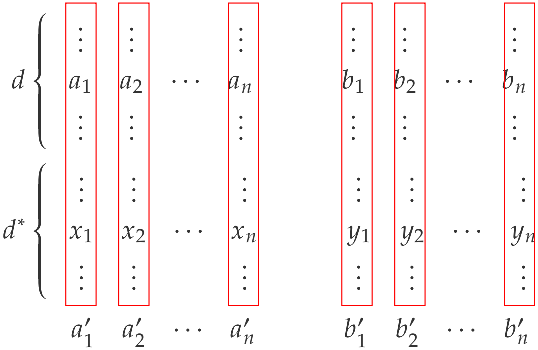
- (n, d, A, B) be instance of Bichromatic Closest Pair in ℓ_p-metric
- $d^* = \mathsf{cd}_p(K_{n,n})$
- $X, Y \subset \mathbb{R}^{d^*}$, $|X| = |Y| = n \longrightarrow$ "polar-pair of point-sets"
- We build $(n, d + d^*, A' \cup B')$ instance of Closest Pair in ℓ_p-metric

Proof Sketch

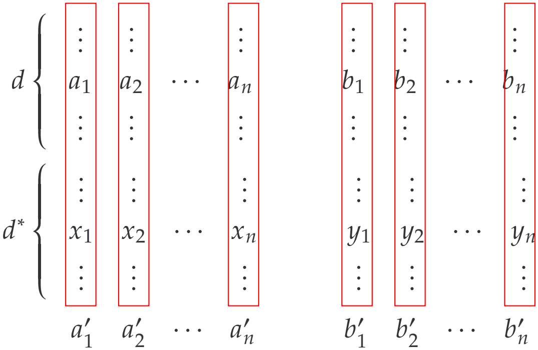
- (n, d, A, B) be instance of Bichromatic Closest Pair in ℓ_p-metric
- $d^* = \mathsf{cd}_p(K_{n,n})$
- $X, Y \subset \mathbb{R}^{d^*}$, $|X| = |Y| = n \longrightarrow$ "polar-pair of point-sets"
- We build $(n, d + d^*, A' \cup B')$ instance of Closest Pair in ℓ_p-metric



Proof Sketch



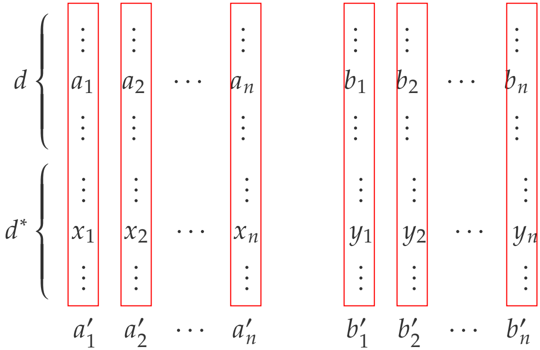
Proof Sketch



Points from same set:

$$\|a'_i - a'_j\|_p^p$$

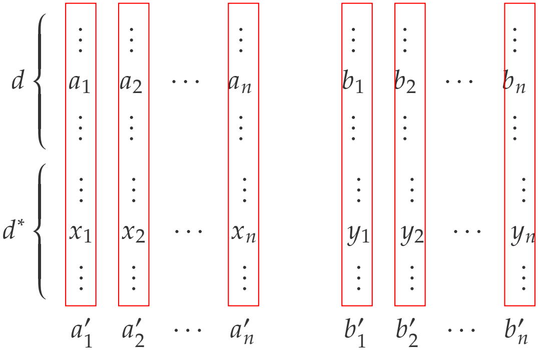
Proof Sketch



Points from same set:

$$\|a'_i - a'_j\|_p^p = \|a_i - a_j\|_p^p + \|x_i - x_j\|_p^p$$

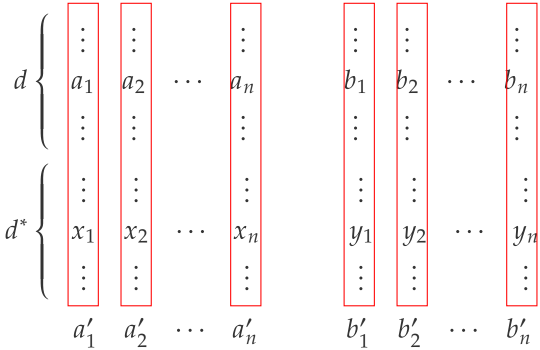
Proof Sketch



Points from same set:

$$\|a'_i - a'_j\|_p^p = \|a_i - a_j\|_p^p + \|x_i - x_j\|_p^p > \|x_i - x_j\|_p^p$$

Proof Sketch



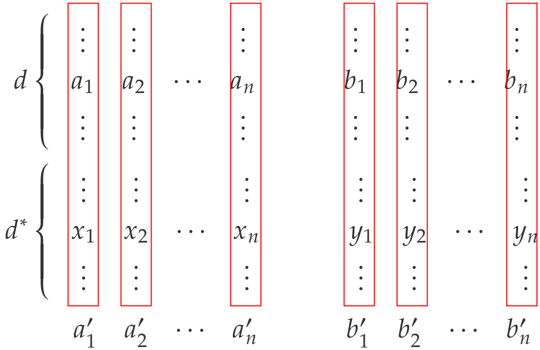
Points from same set:

$$\|a'_i - a'_j\|_p^p = \|a_i - a_j\|_p^p + \|x_i - x_j\|_p^p > \|x_i - x_j\|_p^p$$

Points from different sets:

$$\|a'_i - b'_j\|_p^p$$

Proof Sketch



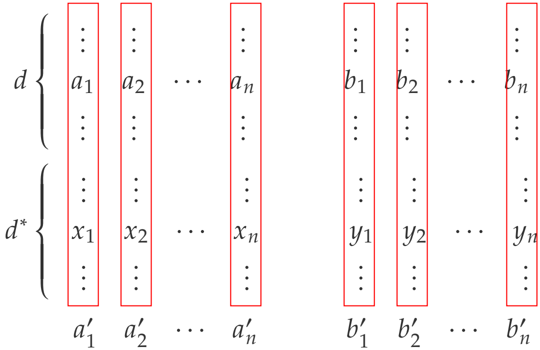
Points from same set:

$$\|a'_i - a'_j\|_p^p = \|a_i - a_j\|_p^p + \|x_i - x_j\|_p^p > \|x_i - x_j\|_p^p$$

Points from different sets:

$$\|a'_i - b'_j\|_p^p = \|a_i - b_j\|_p^p + \|x_i - y'_j\|_p^p$$

Proof Sketch



Points from same set:

$$\|a'_i - a'_j\|_p^p = \|a_i - a_j\|_p^p + \|x_i - x_j\|_p^p > \|x_i - x_j\|_p^p$$

Points from different sets:

$$\|a'_i - b'_j\|_p^p = \|a_i - b_j\|_p^p + \|x_i - y'_j\|_p^p = 1 + \|a_i - b_j\|_p^p$$

Contact Dimension and Sphericity: Table of Results

Metric	Bound	From
ℓ_0	$\mathsf{sph}_0(K_{n,n}) = \mathsf{cd}_0(K_{n,n}) = n$	This paper

Contact Dimension and Sphericity: Table of Results

Metric	Bound	From
ℓ_0	$\mathsf{sph}_0(K_{n,n}) = \mathsf{cd}_0(K_{n,n}) = n$	This paper
ℓ_1	$\Omega(\log n) \leq \mathsf{sph}_1(K_{n,n}) \leq \mathsf{cd}_1(K_{n,n}) \leq n^2$	This paper

Contact Dimension and Sphericity: Table of Results

Metric	Bound	From
ℓ_0	$\mathsf{sph}_0(K_{n,n}) = \mathsf{cd}_0(K_{n,n}) = n$	This paper
ℓ_1	$\Omega(\log n) \leq \mathsf{sph}_1(K_{n,n}) \leq \mathsf{cd}_1(K_{n,n}) \leq n^2$	This paper
$\ell_p, p \in (1,2)$	$\Omega(\log n) \leq \mathsf{sph}_p(K_{n,n}) \leq \mathsf{cd}_1(K_{n,n}) \leq 2n$	This paper

Contact Dimension and Sphericity: Table of Results

Metric	Bound	From
ℓ_0	$\mathsf{sph}_0(K_{n,n}) = \mathsf{cd}_0(K_{n,n}) = n$	This paper
ℓ_1	$\Omega(\log n) \leq \mathsf{sph}_1(K_{n,n}) \leq \mathsf{cd}_1(K_{n,n}) \leq n^2$	This paper
$\ell_p, p \in (1,2)$	$\Omega(\log n) \leq \mathsf{sph}_p(K_{n,n}) \leq \mathsf{cd}_1(K_{n,n}) \leq 2n$	This paper
ℓ_2	$n < \mathsf{sph}_2(K_{n,n}) \leq \mathsf{cd}_2(K_{n,n}) < 1.5 \cdot n$	Maehara'91, Frankl-Maehara'88

Contact Dimension and Sphericity: Table of Results

Metric	Bound	From
ℓ_0	$\mathsf{sph}_0(K_{n,n}) = \mathsf{cd}_0(K_{n,n}) = n$	This paper
ℓ_1	$\Omega(\log n) \leq \mathsf{sph}_1(K_{n,n}) \leq \mathsf{cd}_1(K_{n,n}) \leq n^2$	This paper
$\ell_p, p \in (1,2)$	$\Omega(\log n) \leq \mathsf{sph}_p(K_{n,n}) \leq \mathsf{cd}_1(K_{n,n}) \leq 2n$	This paper
ℓ_2	$n < \mathsf{sph}_2(K_{n,n}) \leq \mathsf{cd}_2(K_{n,n}) < 1.5 \cdot n$	Maehara'91, Frankl-Maehara'88
$\ell_p, p > 2$	$\mathsf{sph}_p(K_{n,n}) = \Theta(\mathsf{cd}_p(K_{n,n})) = \Theta(\log n)$	This paper

Contact Dimension and Sphericity: Table of Results

Metric	Bound	From
ℓ_0	$\mathsf{sph}_0(K_{n,n}) = \mathsf{cd}_0(K_{n,n}) = n$	This paper
ℓ_1	$\Omega(\log n) \leq \mathsf{sph}_1(K_{n,n}) \leq \mathsf{cd}_1(K_{n,n}) \leq n^2$	This paper
$\ell_p, p \in (1,2)$	$\Omega(\log n) \leq \mathsf{sph}_p(K_{n,n}) \leq \mathsf{cd}_1(K_{n,n}) \leq 2n$	This paper
ℓ_2	$n < \mathsf{sph}_2(K_{n,n}) \leq \mathsf{cd}_2(K_{n,n}) < 1.5 \cdot n$	Maehara'91, Frankl-Maehara'88
$\ell_p, p > 2$	$\mathsf{sph}_p(K_{n,n}) = \Theta(\mathsf{cd}_p(K_{n,n})) = \Theta(\log n)$	This paper
ℓ_∞	$\mathsf{sph}_\infty(K_{n,n}) = \mathsf{cd}_\infty(K_{n,n}) = 2\log_2 n$	Roberts'69

SETH Lower Bound for Closest Pair

Theorem (Alman-Williams'15)

For any $\varepsilon > 0$, $p \in \mathbb{R}_{\geq 1} \cup \{0\}$, and $d = \omega(\log n)$, Bichromatic Closest Pair in ℓ_p-metric admits no $(n^{2-\varepsilon})$-time algorithm unless SETH is false.

SETH Lower Bound for Closest Pair

Theorem (Alman-Williams'15)

For any $\varepsilon > 0$, $p \in \mathbb{R}_{\geq 1} \cup \{0\}$, and $d = \omega(\log n)$, Bichromatic Closest Pair in ℓ_p-metric admits no $(n^{2-\varepsilon})$-time algorithm unless SETH is false.

For $p > 2$, we have $\mathsf{cd}_p(K_{n,n}) = \Theta(\log n)$

SETH Lower Bound for Closest Pair

Theorem (Alman-Williams'15)

For any $\varepsilon > 0$, $p \in \mathbb{R}_{\geq 1} \cup \{0\}$, and $d = \omega(\log n)$, Bichromatic Closest Pair in ℓ_p-metric admits no $(n^{2-\varepsilon})$-time algorithm unless SETH is false.

For $p > 2$, we have $\mathsf{cd}_p(K_{n,n}) = \Theta(\log n)$

Theorem (Our Result)

For any $\varepsilon > 0$, $p > 2$, and $d = \omega(\log n)$, Closest Pair in ℓ_p-metric admits no $(n^{2-\varepsilon})$-time algorithm unless SETH is false.

SETH Lower Bound for approximate Closest Pair

Theorem (Rubinstein'18)

For any $\varepsilon > 0$, $p \in \mathbb{R}_{\geq 1} \cup \{0\}$, *and* $d = \omega(\log n)$, *there exists* $\delta(\varepsilon, p)$ *such that Bichromatic Closest Pair in* ℓ_p*-metric admits no* $(n^{2-\varepsilon})$*-time* $(1 + \delta)$*-approximation algorithm unless SETH is false.*

SETH Lower Bound for approximate Closest Pair

Theorem (Rubinstein'18)

For any $\varepsilon > 0$, $p \in \mathbb{R}_{\geq 1} \cup \{0\}$, and $d = \omega(\log n)$, there exists $\delta(\varepsilon, p)$ such that Bichromatic Closest Pair in ℓ_p-metric admits no $(n^{2-\varepsilon})$-time $(1+\delta)$-approximation algorithm unless SETH is false.

Theorem (Our Result)

For any $\varepsilon > 0$, $p > 2$, and $d = \omega(\log n)$, there exists $\delta(\varepsilon, p)$ such that Closest Pair in ℓ_p-metric admits no $(n^{2-\varepsilon})$-time $(1+\delta)$-approximation algorithm unless SETH is false.

Proof of $\mathsf{cd}_p(K_{n,n}) = \Theta(\log n)$ for $p > 2$

- $\mathsf{cd}_p(K_{n,n}) = \Omega(\log n)$ for $p \in \mathbb{R}_{\geq 1} \cup \{0\}$
  - Sphere packing bound

Proof of $\mathsf{cd}_p(K_{n,n}) = \Theta(\log n)$ for $p > 2$

- $\mathsf{cd}_p(K_{n,n}) = \Omega(\log n)$ for $p \in \mathbb{R}_{\geq 1} \cup \{0\}$
  - Sphere packing bound
- $\mathsf{cd}_p(K_{n,n}) = O(\log n)$ for $p > 2$
  - Let $C \subseteq \{0,1\}^{O(d)}$ where $|C| = n$ and $\forall c, c' \in C, \|c - c'\|_0 \geq \delta$

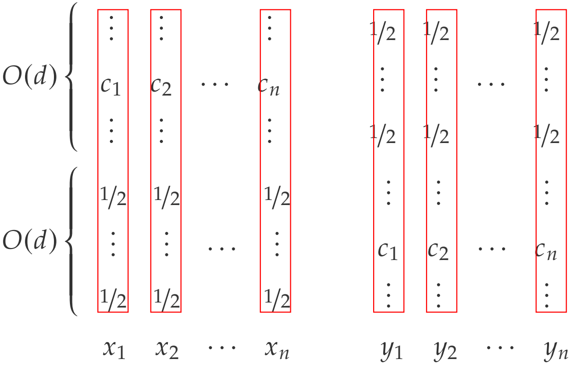
Proof of $\mathrm{cd}_p(K_{n,n}) = \Theta(\log n)$ for $p > 2$

- $\mathrm{cd}_p(K_{n,n}) = \Omega(\log n)$ for $p \in \mathbb{R}_{\geq 1} \cup \{0\}$
  - Sphere packing bound
- $\mathrm{cd}_p(K_{n,n}) = O(\log n)$ for $p > 2$
  - Let $C \subseteq \{0,1\}^{O(d)}$ where $|C| = n$ and $\forall c, c' \in C, \|c - c'\|_0 \geq \delta$

$$
\begin{array}{r}
O(d)\left\{\begin{array}{cccc} \vdots & \vdots & & \vdots \\ c_1 & c_2 & \cdots & c_n \\ \vdots & \vdots & & \vdots \end{array}\right. \\
O(d)\left\{\begin{array}{cccc} 1/2 & 1/2 & & 1/2 \\ \vdots & \vdots & \cdots & \vdots \\ 1/2 & 1/2 & & 1/2 \end{array}\right.
\end{array}
\quad
\begin{array}{cccc} 1/2 & 1/2 & & 1/2 \\ \vdots & \vdots & \cdots & \vdots \\ 1/2 & 1/2 & & 1/2 \\ \vdots & \vdots & & \vdots \\ c_1 & c_2 & \cdots & c_n \\ \vdots & \vdots & & \vdots \end{array}
$$

Proof of $\text{cd}_p(K_{n,n}) = \Theta(\log n)$ for $p > 2$

- $\text{cd}_p(K_{n,n}) = \Omega(\log n)$ for $p \in \mathbb{R}_{\geq 1} \cup \{0\}$
  - Sphere packing bound
- $\text{cd}_p(K_{n,n}) = O(\log n)$ for $p > 2$
  - Let $C \subseteq \{0,1\}^{O(d)}$ where $|C| = n$ and $\forall c, c' \in C, \|c - c'\|_0 \geq \delta$



Contact Dimension and Sphericity: Table of Results

Metric	Bound	From
ℓ_0	$\mathsf{sph}_0(K_{n,n}) = \mathsf{cd}_0(K_{n,n}) = n$	This paper
ℓ_1	$\Omega(\log n) \leq \mathsf{sph}_1(K_{n,n}) \leq \mathsf{cd}_1(K_{n,n}) \leq n^2$	This paper
$\ell_p, p \in (1,2)$	$\Omega(\log n) \leq \mathsf{sph}_p(K_{n,n}) \leq \mathsf{cd}_1(K_{n,n}) \leq 2n$	This paper
ℓ_2	$n < \mathsf{sph}_2(K_{n,n}) \leq \mathsf{cd}_2(K_{n,n}) < 1.5 \cdot n$	Maehara'91, Frankl-Maehara'88
$\ell_p, p > 2$	$\mathsf{sph}_p(K_{n,n}) = \Theta(\mathsf{cd}_p(K_{n,n})) = \Theta(\log n)$	This paper
ℓ_∞	$\mathsf{sph}_\infty(K_{n,n}) = \mathsf{cd}_\infty(K_{n,n}) = 2\log_2 n$	Roberts'69

Curious Case of ℓ_1-metric

Theorem

For any integer $d > 0$, there exist no two finitely supported random variables X, Y taking values from $\mathbb{R}^d$ such that the following hold.

$$\mathop{\mathbb{E}}_{x_1, x_2 \in_R X}\left[\|x_1 - x_2\|_1\right] > \mathop{\mathbb{E}}_{x_1 \in_R X,\, y_1 \in_R Y}\left[\|x_1 - y_1\|_1\right]$$

$$\mathop{\mathbb{E}}_{y_1, y_2 \in_R Y}\left[\|y_1 - y_2\|_1\right] > \mathop{\mathbb{E}}_{x_1 \in_R X,\, y_1 \in_R Y}\left[\|x_1 - y_1\|_1\right]$$

Curious Case of ℓ_1-metric

Theorem

For any integer $d > 0$, there exist no two finitely supported random variables X, Y taking values from $\mathbb{R}^d$ such that the following hold.

$$\mathop{\mathbb{E}}_{x_1, x_2 \in_R X}\left[\|x_1 - x_2\|_1\right] > \mathop{\mathbb{E}}_{x_1 \in_R X,\, y_1 \in_R Y}\left[\|x_1 - y_1\|_1\right]$$

$$\mathop{\mathbb{E}}_{y_1, y_2 \in_R Y}\left[\|y_1 - y_2\|_1\right] > \mathop{\mathbb{E}}_{x_1 \in_R X,\, y_1 \in_R Y}\left[\|x_1 - y_1\|_1\right]$$

Corollary

For any $\alpha > 0$, there exist no subsets $A, B \subset \mathbb{R}^d$ of $n/2$ vectors with $d < n/2$ such that

- *For any u, v both in A, or both in B, $\|u - v\|_1 \geq \frac{1}{1-2/n} \cdot \alpha$.*
- *For any $u \in A$ and $v \in B$, $\|u - v\|_1 < \alpha$.*

Open Problems

- Complexity of Closest Pair in ℓ_2, ℓ_1, ℓ_0-metrics when $d \approx \log n$?

Open Problems

- Complexity of Closest Pair in ℓ_2, ℓ_1, ℓ_0-metrics when $d \approx \log n$?
    - Need white-box techniques?

Open Problems

- Complexity of Closest Pair in ℓ_2, ℓ_1, ℓ_0-metrics when $d \approx \log n$?
  - Need white-box techniques?
- Is $\mathsf{cd}_1(K_{n,n}) = \Omega(n)$?

Open Problems

- Complexity of Closest Pair in ℓ_2, ℓ_1, ℓ_0-metrics when $d \approx \log n$?
  - Need white-box techniques?
- Is $\mathsf{cd}_1(K_{n,n}) = \Omega(n)$?
  - Closely related to Kusner's conjecture: $\mathsf{cd}_1(K_n) = n/2$
  - Alon-Pudlák'03: $\mathsf{cd}_1(K_n) = \Omega(n/\log n)$
- Is $\mathsf{cd}_0(K_{n,n}) = O(n)$ over $\{0, 1\}$?

Open Problems

- Complexity of Closest Pair in ℓ_2, ℓ_1, ℓ_0-metrics when $d \approx \log n$?
  - Need white-box techniques?

- Is $\mathsf{cd}_1(K_{n,n}) = \Omega(n)$?
  - Closely related to Kusner's conjecture: $\mathsf{cd}_1(K_n) = n/2$
  - Alon-Pudlák'03: $\mathsf{cd}_1(K_n) = \Omega(n/\log n)$

- Is $\mathsf{cd}_0(K_{n,n}) = O(n)$ over $\{0,1\}$?
  - We showed: $\mathsf{cd}_0(K_{n,n})$ over $\{0,1\}$ is in $[n, n^2]$

Key Takeaways

- Closest Pair problem is hard in ℓ_p-metric when $d = \omega(\log n)$ and $p > 2$

Key Takeaways

- Closest Pair problem is hard in ℓ_p-metric when $d = \omega(\log n)$ and $p > 2$– even to approximate!

Key Takeaways

- Closest Pair problem is hard in ℓ_p-metric when $d = \omega(\log n)$ and $p > 2$– even to approximate!

- Closest Pair and Bichromatic Closest Pair are equivalent when $d = \omega\left(\mathsf{cd}\left(K_{n,n}\right)\right)$

Key Takeaways

- Closest Pair problem is hard in ℓ_p-metric when $d = \omega(\log n)$ and $p > 2$– even to approximate!
- Closest Pair and Bichromatic Closest Pair are equivalent when $d = \omega\left(\mathsf{cd}\left(K_{n,n}\right)\right)$
- Complexity of Closest Pair problem in Euclidean metric for $d \approx \log n$ – open!

THANK
YOU!