Ham Sandwich is Equivalent to Borsuk-Ulam

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Weizmann Institute of Science

July 4^{th} 2017

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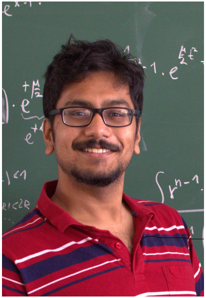
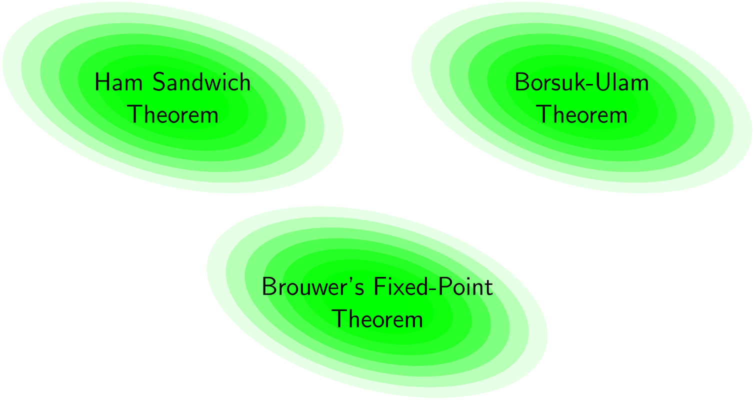
Joint work with Arpan Saha

(University of Hamburg)



Fixed-Point Theorems and Computation



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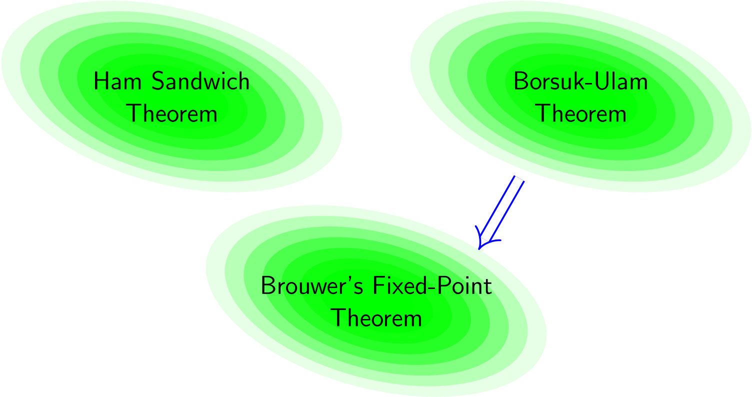

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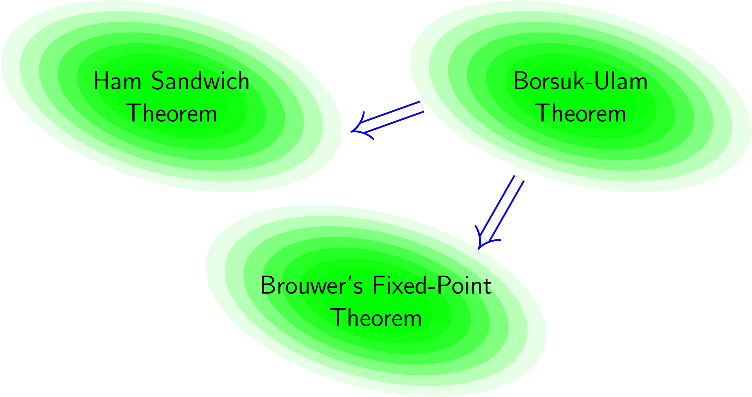

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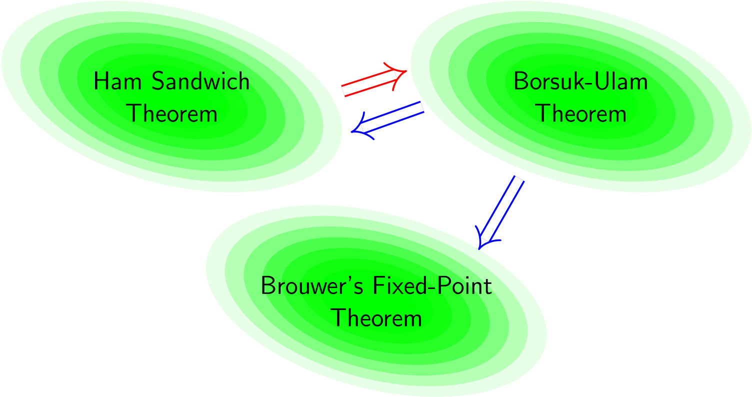

Borsuk-Ulam Theorem

Theorem (Borsuk, 1933)

Let S^n denote the set of all points on the unit n-dimensional sphere. For any odd continuous mapping $f : S^n \to \mathbb{R}^n$ there is a point $x \in S^n$ for which $f(x) = \vec{0}$.

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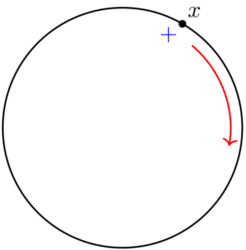

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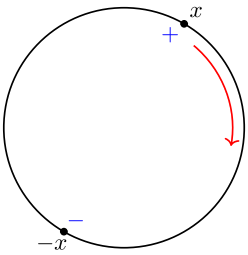

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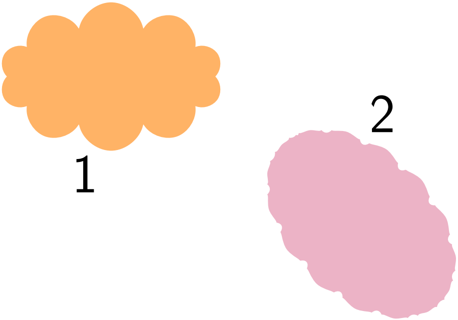

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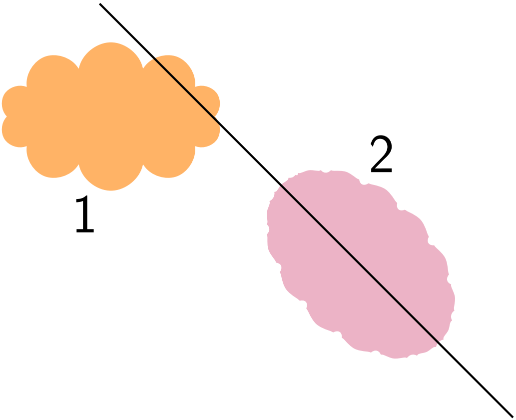

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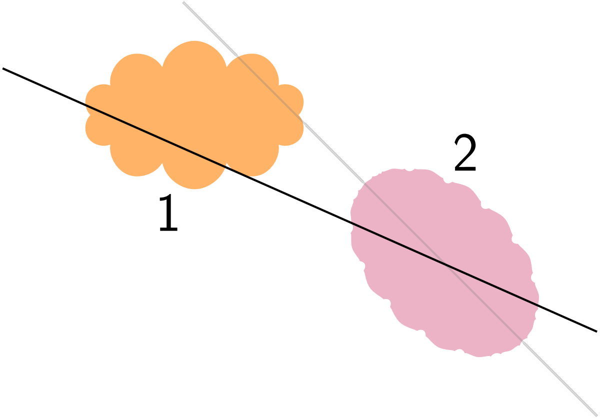

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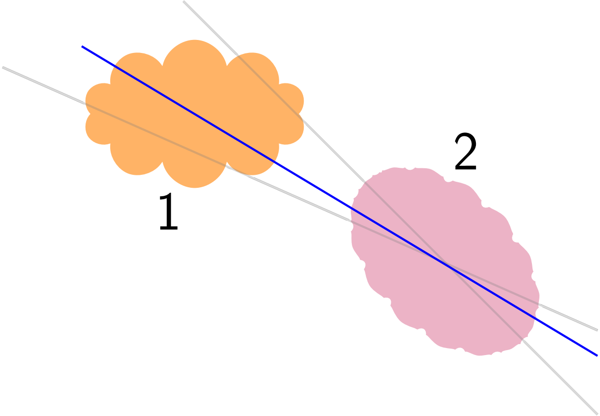
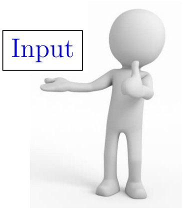


Borsuk-Ulam $\iff$ Ham Sandwich

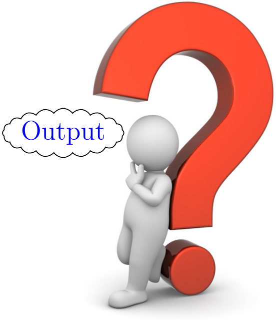
Theorem (Our Result)

Ham Sandwich theorem is equivalent to Borsuk-Ulam theorem.

Query Model

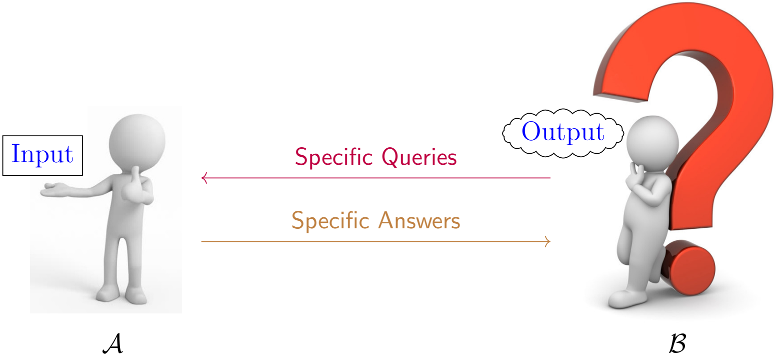


$\mathcal{A}$



$\mathcal{B}$

Query Model


Input
Output
Specific Queries
Specific Answers
$\mathcal{A}$
$\mathcal{B}$

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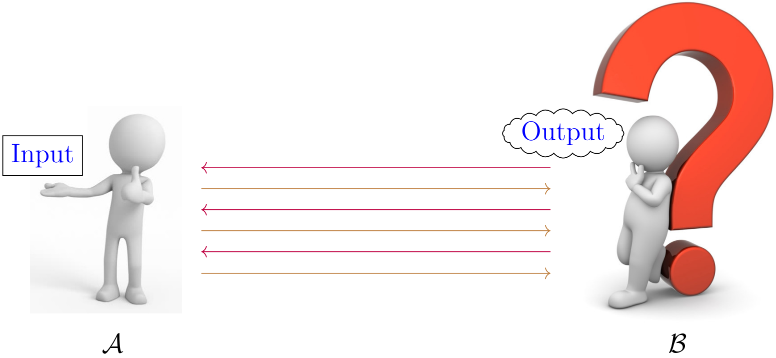

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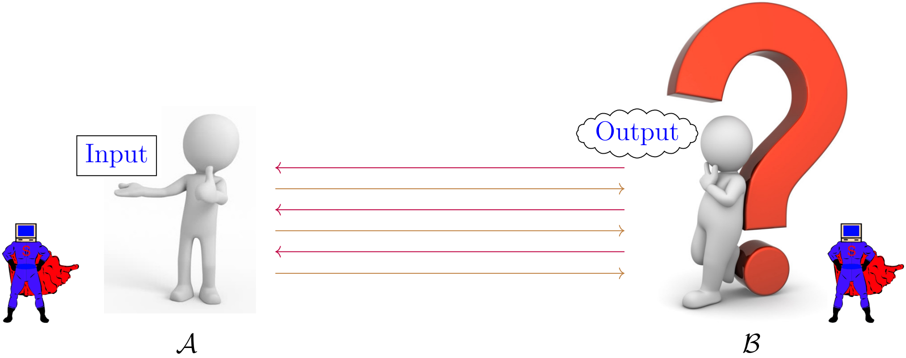

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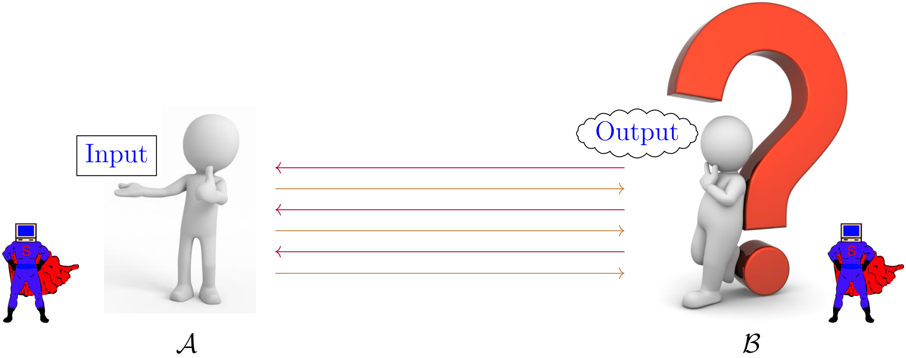

QC_p: Number of queries to find correct answer with probability p.

Ham Sandwich Problem

$\mathrm{ABH}(n, k, \varepsilon)$ Problem:

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From Rubinstein (2016), Su's construction (1997), and Ham Sandwich theorem $\Rightarrow$ Borsuk-Ulam theorem:

Theorem (Our Result)

For large n, $\varepsilon \leq 1/\mathit{poly}(n)$, and for $p = 2^{-\Omega(n)}$ and $k \geq 5$ we have:
$$QC_p(ABH(n, k, \varepsilon)) = 2^{\Omega(n)}.$$

Borsuk-Ulam $\implies$ Ham Sandwich

- Given $A_1, \ldots, A_n, A_{n+1}$ compact sets in $\mathbb{R}^{n+1}$
- Build odd $f : S^n \to \mathbb{R}^n$ such that:

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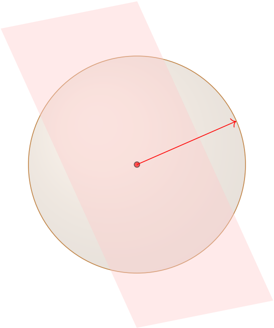

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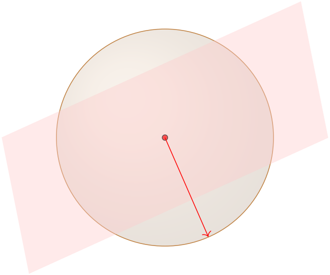

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- For every $x \in S^n$:

$$\boxed{f_i(x) = \mathsf{vol}(A_i \cap H_x^+) - \mathsf{vol}(A_i \cap H_x^-)}$$

Our Result: Borsuk-Ulam $\Longleftarrow$ Ham Sandwich

Observation (From Previous Proof)

Let A be a compact set in $\mathbb{R}^{n+1}$. Then, there is a continuous odd function $f : S^n \to \mathbb{R}$ such that $\forall x \in S^n,\ f(x) = \text{vol}(A \cap H_x^+) - \text{vol}(A \cap H_x^-)$.

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Conjecture (Wishful Thinking)

Let $f : S^n \to \mathbb{R}$ be a continuous odd function. Then, there is a compact set A in $\mathbb{R}^{n+1}$ such that $\forall x \in S^n,\ f(x) = \text{vol}(A \cap H_x^+) - \text{vol}(A \cap H_x^-)$.

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$$A_r \subset \mathbb{R}^{n+1}$$

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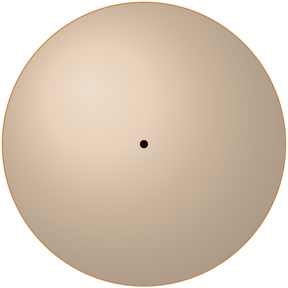

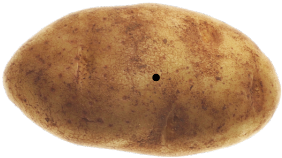

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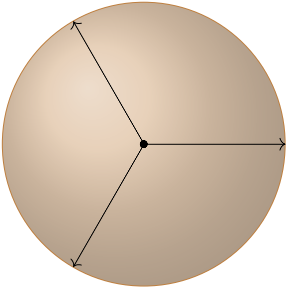

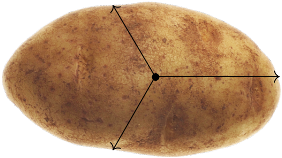

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$$f = \sum_i \beta_i \cdot p_i$$

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Proof (continued). Let $p_1, p_2, \ldots, p_{m(d)}$ be a basis of polynomials of degree d over the hypersphere. If $r' = \sum_i \alpha_i \cdot p_i$ then we have:

$$\text{vol}(A_r \cap H_x^+) - \text{vol}(A_r \cap H_x^-) = \sum_i \left(\alpha_i \cdot \int_{y \in S^n} \text{sgn}(\langle x, y \rangle) \cdot p_i(y)\ \mathrm{d}y \right)$$

$$f = \sum_i \beta_i \cdot p_i$$

We need to find a basis such that:

$$\boxed{\int_{y \in S^n} \text{sgn}(\langle x, y \rangle) \cdot p_i(y)\ \mathrm{d}y = \lambda_i \cdot p_i(x)}$$

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- For every polynomial p there is a hyperspherical harmonic h such that $p|_{S^n} = h$ (Folklore, 1800s)
- Eigen functions of the following operator T (Funk and Hecke, 1917):

$$T(f)(x) := \int_{y \in S^n} u(\langle x, y \rangle) \cdot f(y) \, \mathrm{d}y,$$

where $u : [-1, 1] \to \mathbb{R}$ is bounded and measurable

Key Takeaways

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- Borsuk-Ulam Theorem is Equivalent to Ham Sandwich Theorem!
- Ham Sandwich Problem in high dimensions is Hard!

Thank you!