An Efficient Representation for Filtrations of Simplicial Complexes

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Joint work with Jean-Daniel Boissonnat (INRIA).

Simplicial Complex (definition)

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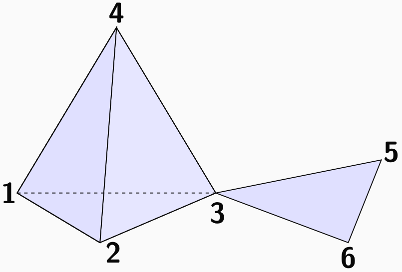

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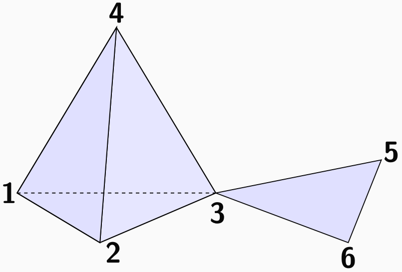

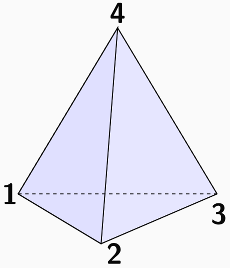

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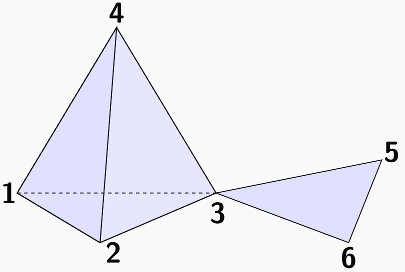

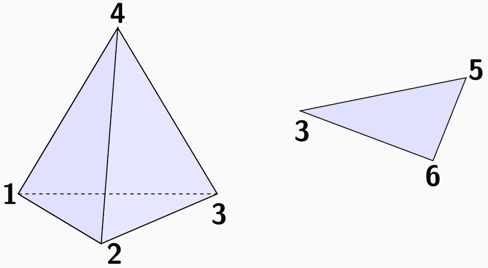

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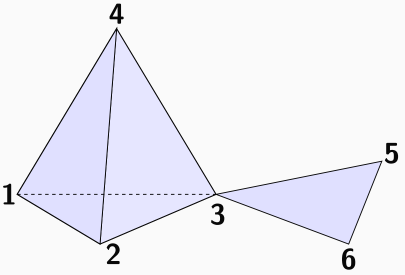

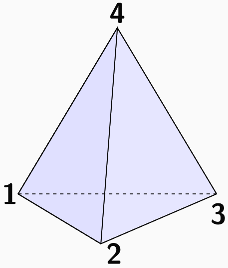

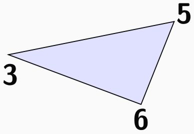

6 vertices
3 dimensional
2 maximal simplices
22 simplices

Data structures for simplicial complexes:

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⋆ Hasse Diagram

Representation of Simplicial Complexes

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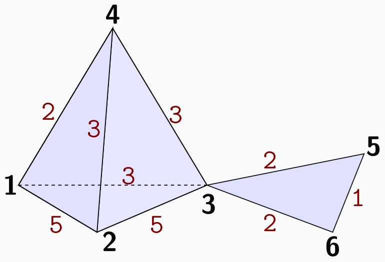

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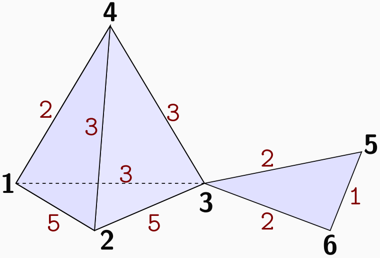

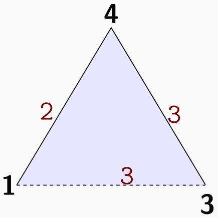

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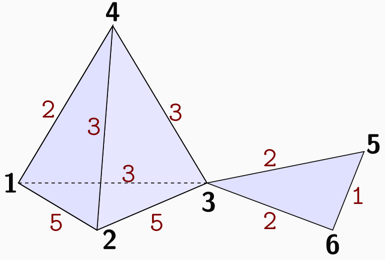

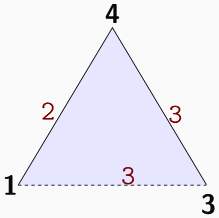

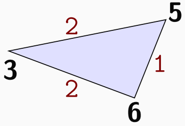

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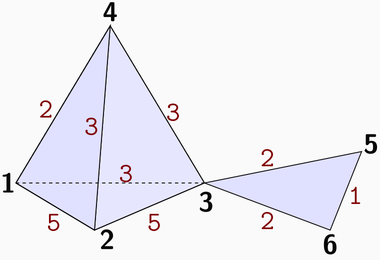

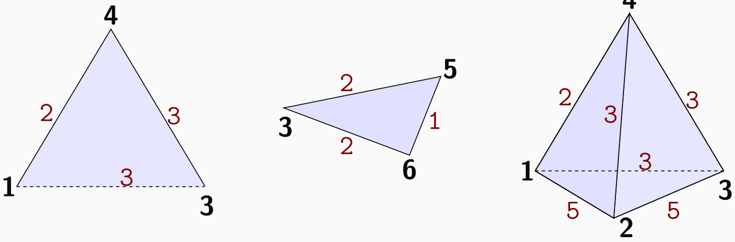

Agenda

Find a representation for filtrations of simplicial complexes:

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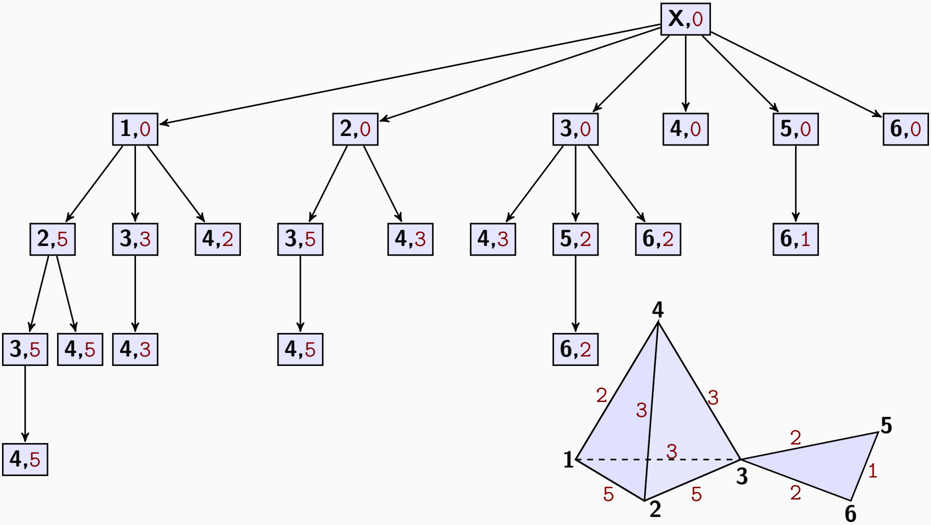
Find a representation for filtrations of simplicial complexes:

- ⋆ Small size.
- ⋆ Perform queries quickly:
  - Access filtration value.
  - Simplex Insertion.
  - Simplex Removal.

Simplex Tree

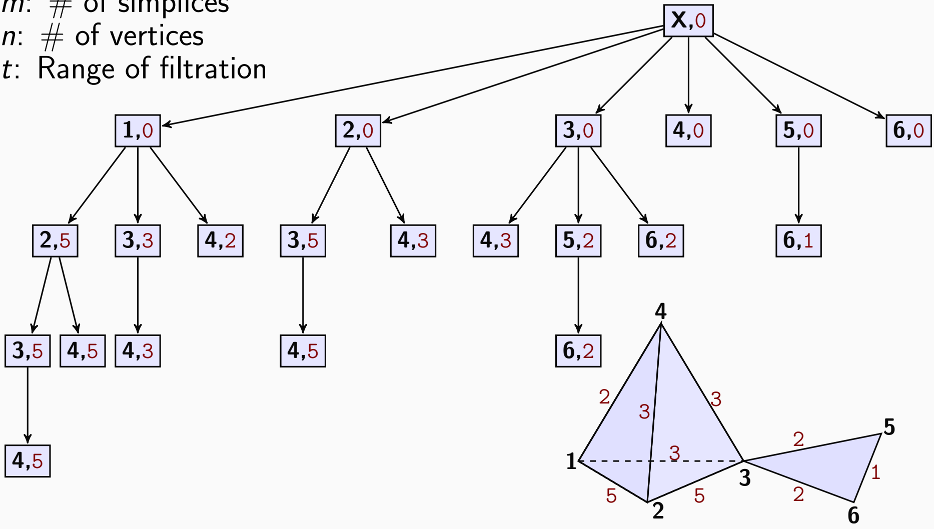
Introduced by Boissonnat and Maria [ESA '12, Algorithmica '14].



Simplex Tree

Storage: $\Theta\left(m\left(\log n + \log t\right)\right)$

m: # of simplices
n: # of vertices
t: Range of filtration



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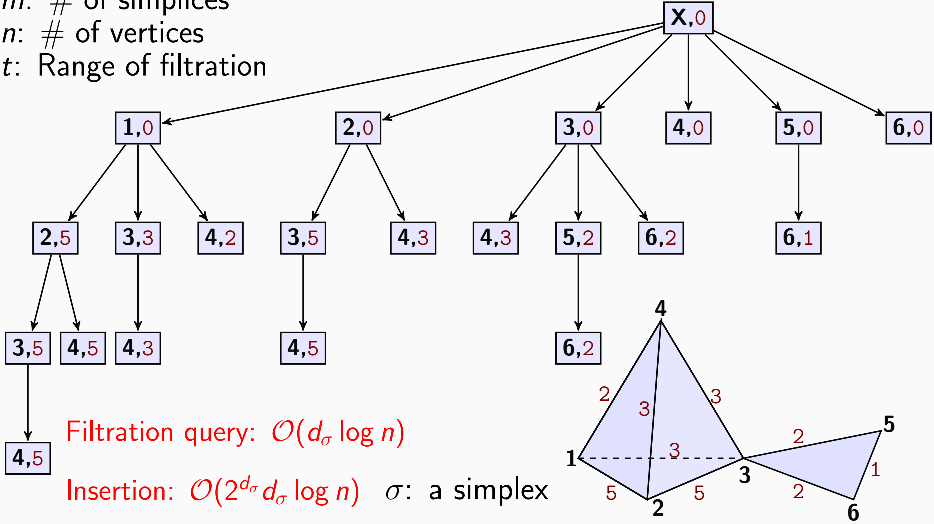

Filtration query: $\mathcal{O}(d_\sigma \log n)$

Insertion: $\mathcal{O}(2^{d_\sigma} d_\sigma \log n)$ σ: a simplex

Removal: $\mathcal{O}(m \log n)$ d_σ: dimension of σ

A Change in Perspective

A simplicial complex $K : \{0,1\}^n \to \{0,1\}$:

- $|x| \leq 1 \implies K(x) = 0$.
- Monotonicity: A directed path from y to $x \Rightarrow K(y) \leq K(x)$.

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A compact representation: store only the boundary!

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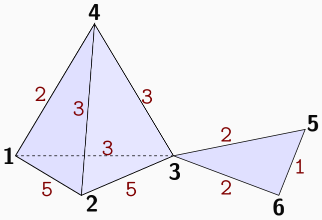
Demonstration



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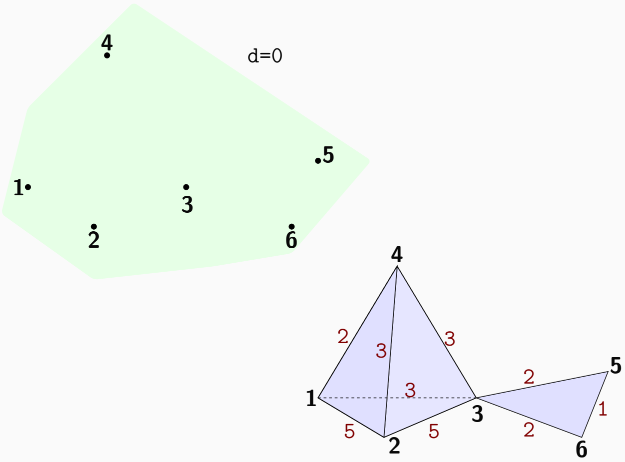

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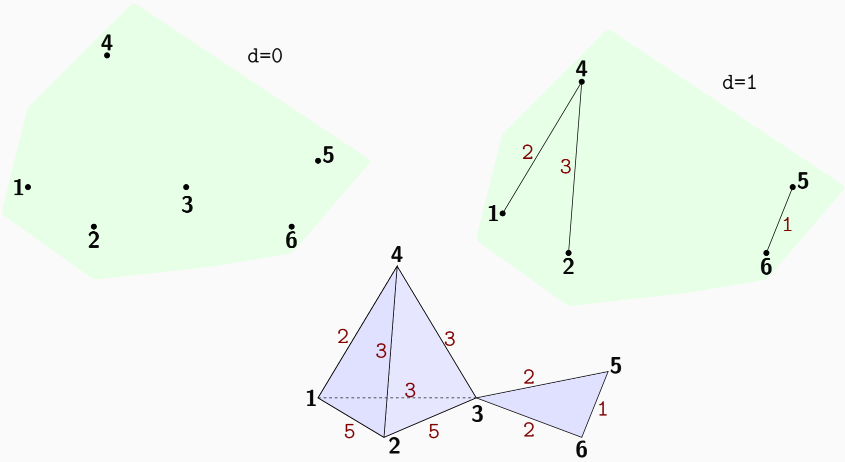
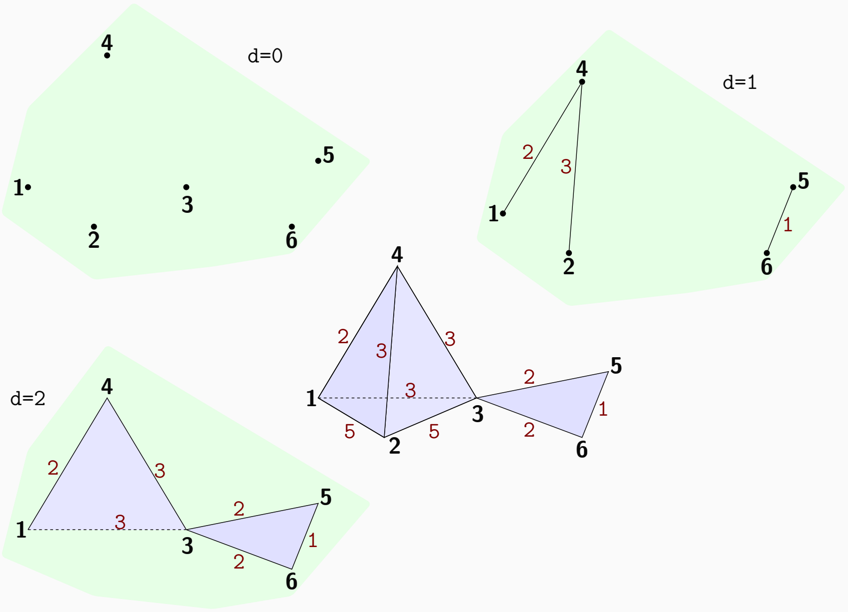
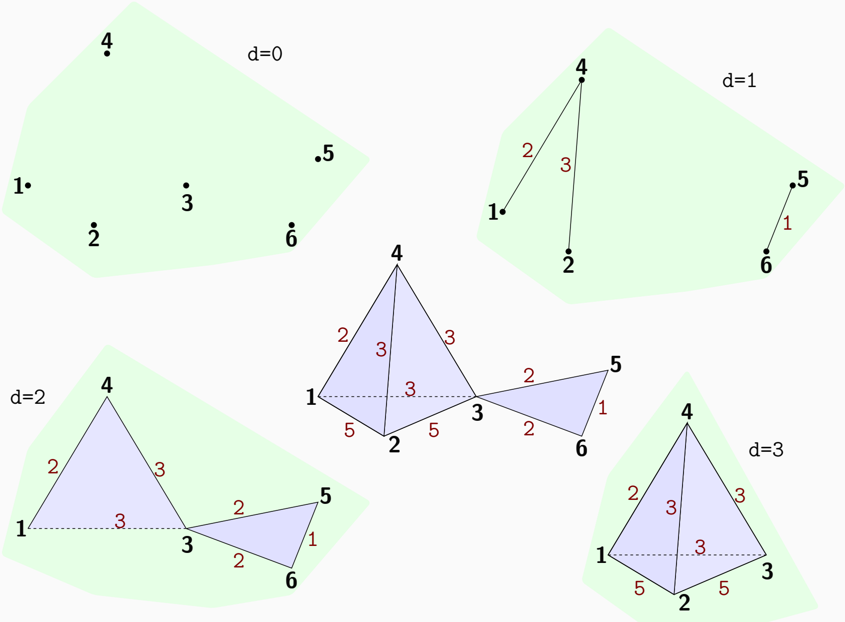



d=0
4
5
1
3
2
6
d=1
4
2
3
5
1
1
2
6
4
2
3
3
3
5
2
1
1
3
5
5
2
2
6
d=2
4
2
3
2
5
1
3
1
3
2
6

Demonstration



Lower Bound on Data Structures

Theorem

Consider the class of all simplicial complexes on n vertices of dimension d, associated with a filtration over the range of $\{0, \ldots, t\}$, such that the number of critical simplices is κ, where $d \geq 2$ and $\kappa \geq n+1$, and consider any data structure that can represent the simplicial complexes of this class. Such a data structure requires $\log \left(\binom{\binom{n/2}{d+1}}{\kappa - n} t^{\kappa - n} \right)$ bits to be stored. For any constant $\varepsilon \in (0,1)$ and for $\frac{2}{\varepsilon} n \leq \kappa \leq n^{(1-\varepsilon)d}$ and $d \leq n^{\varepsilon/3}$, the bound becomes $\Omega(\kappa(d \log n + \log t))$.

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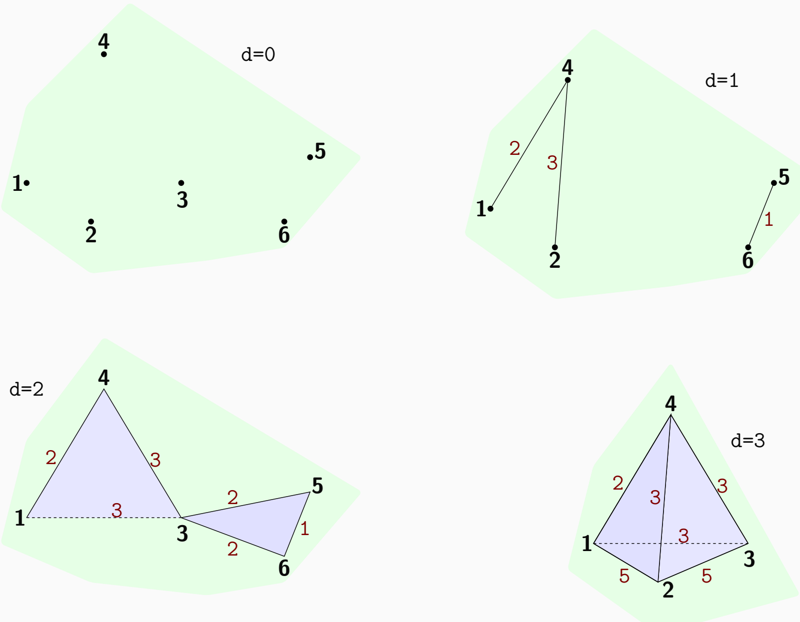
Construction of CSD



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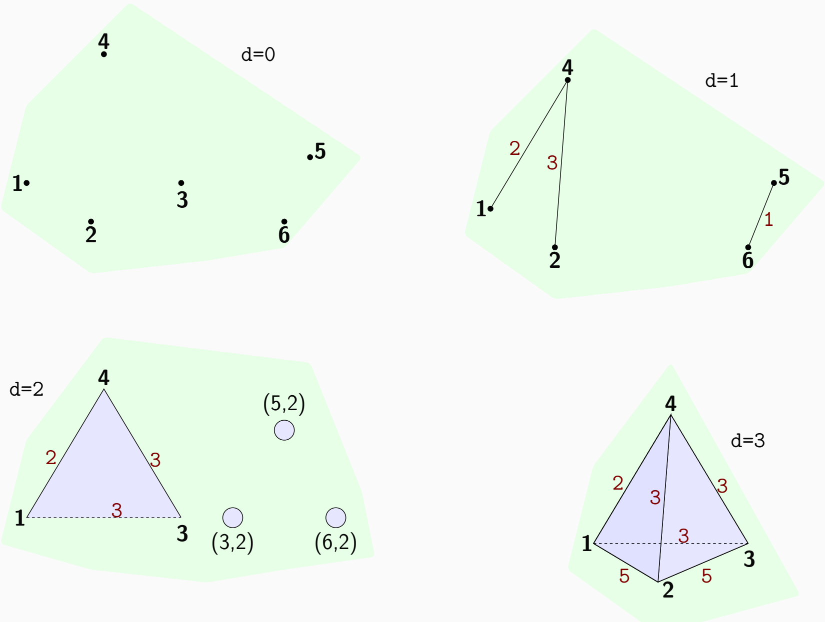

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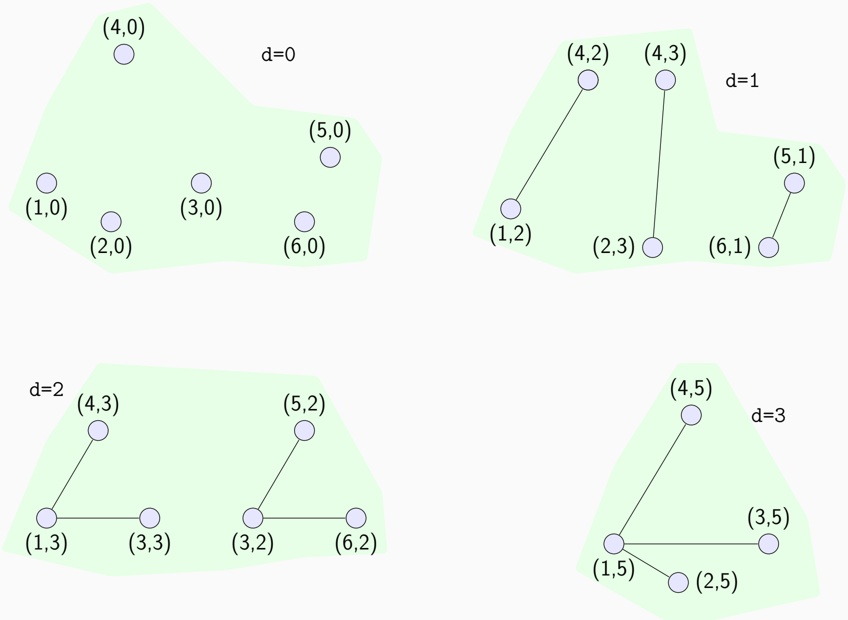

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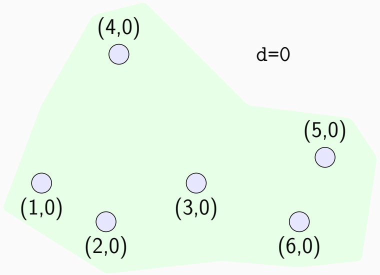

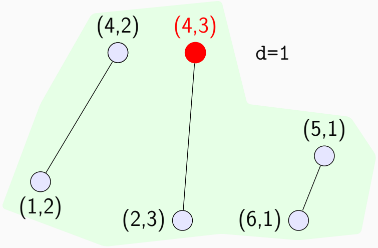

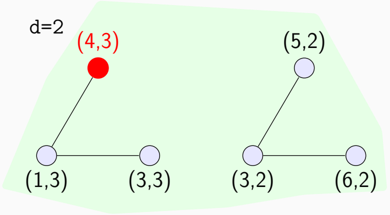

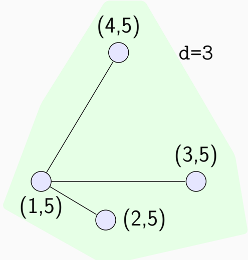

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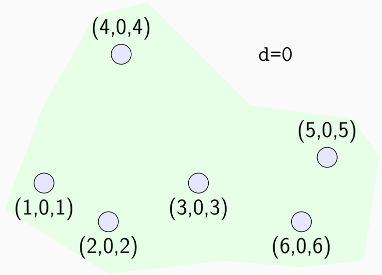

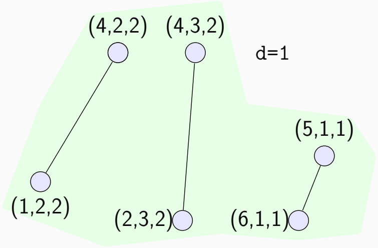

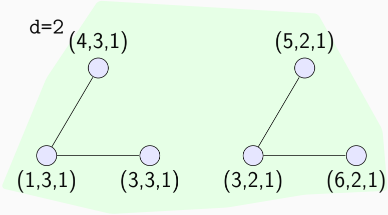

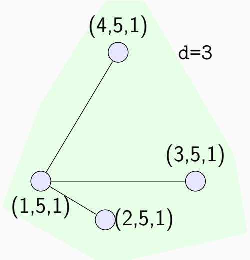

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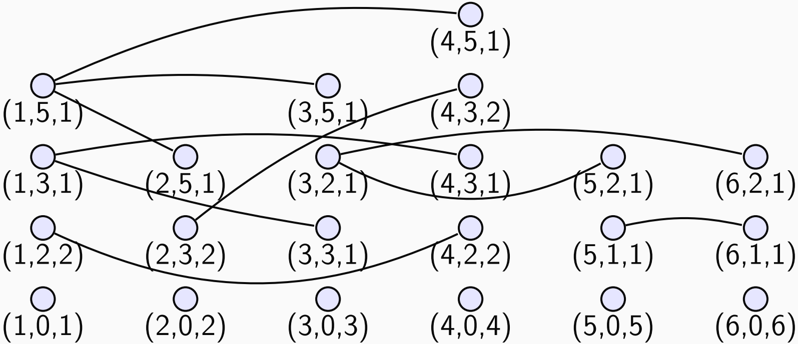

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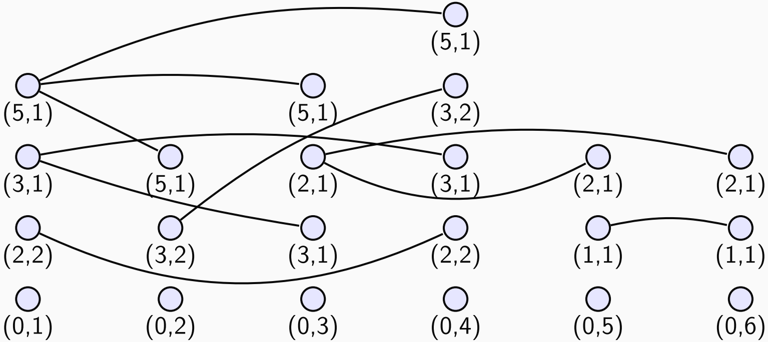

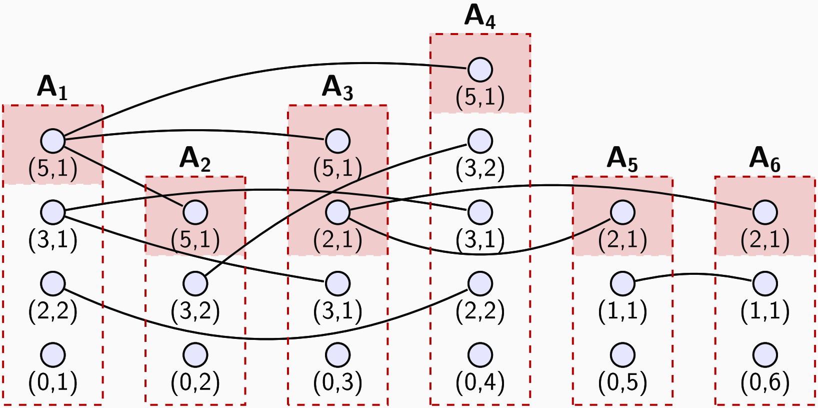
A1
(5,1)
(3,1)
(2,2)
(0,1)
A2
(5,1)
(3,2)
(0,2)
A3
(5,1)
(2,1)
(3,1)
(0,3)
A4
(5,1)
(3,2)
(3,1)
(2,2)
(0,4)
A5
(2,1)
(1,1)
(0,5)
A6
(2,1)
(1,1)
(0,6)

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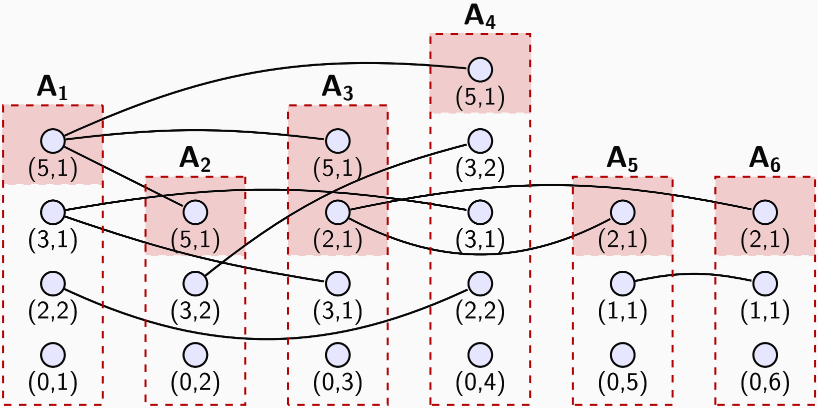

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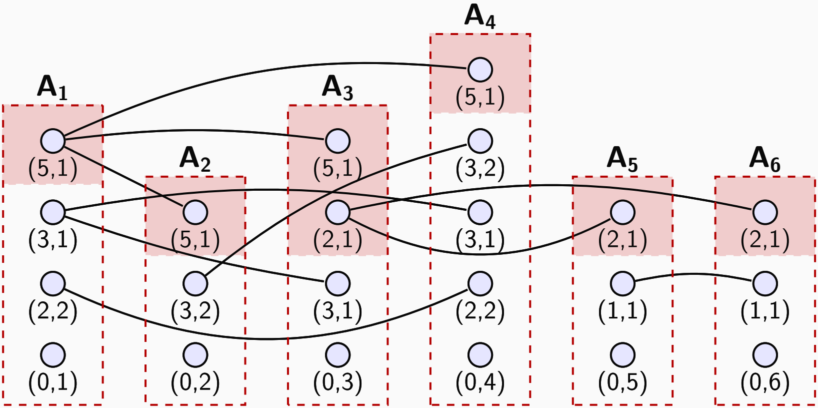
$\kappa :=$ Number of critical simplices



Storage$= \mathcal{O}(\kappa d \log(\kappa t)) \longrightarrow$ "Tight!"

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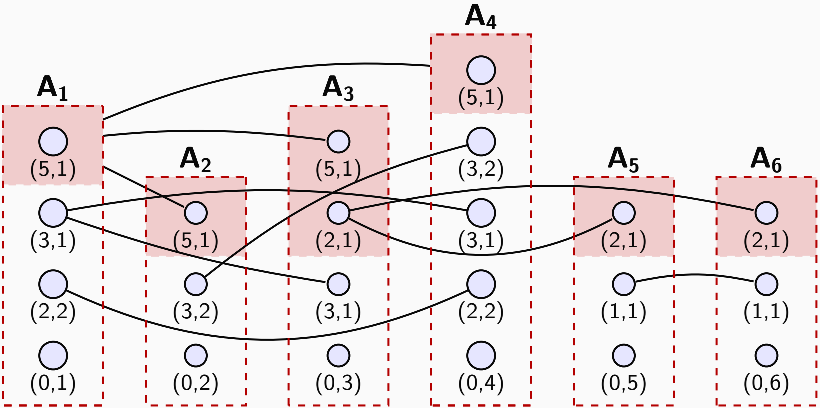
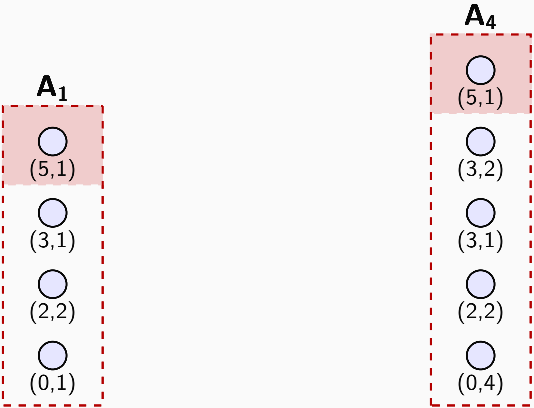
Computing Filtration in CSD



Computing filtration value of $1 - 4$

Computing Filtration in CSD



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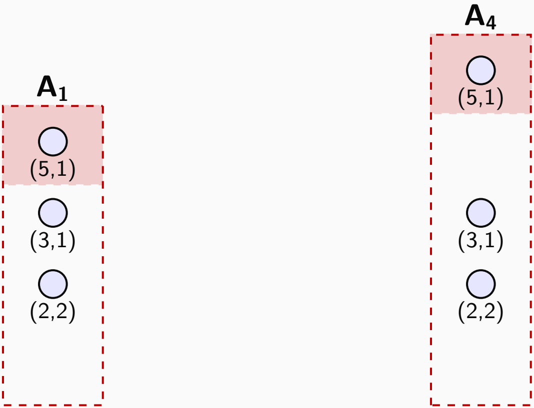

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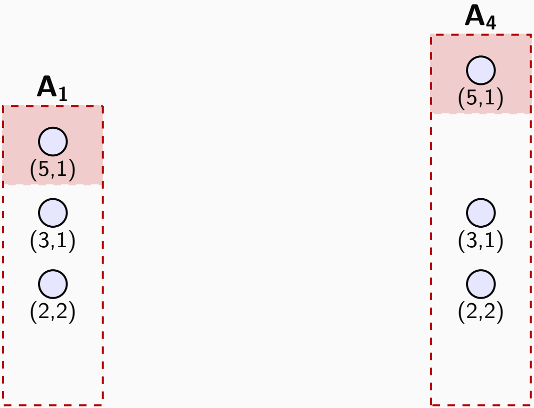

Filtration value of $1-4$

$= \min(2,3,5) = 2$

Computing filtration value of $1-4$

Some Relevant Parameters

Γ_j: largest number of maximal simplices of K that a given j-simplex of K may be contained in.

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Ψ is *mostly* much smaller than κ.

Computing Filtration in CSD

Input: A simplex $\sigma = v_{\ell_0} \cdots v_{\ell_{d_\sigma}}$.

Task: Compute $f(\sigma)$.

1. Find $A_{\ell_0}, \ldots, A_{\ell_{d_\sigma}}$.
2. Compute $A_\sigma = \bigcap_{0 \leq i \leq d_\sigma} A_{\ell_i}$.
3. Let $\mathcal{P}$ be projection on to the first coordinate.

$$f(\sigma) = \min \{\mathcal{P}(x) \mid x \in A_\sigma\}.$$

$$\boxed{\mathcal{O}\left(\Psi d_\sigma \log \Psi\right)}$$

Insertion in CSD

Input: A simplex $\sigma = v_{\ell_0} \cdots v_{\ell_{d_\sigma}}$ with filtration value s_σ.

Task: Insert σ in K with filtration value s_σ.

Insertion in CSD

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   1.1 Remove or relocate all faces of σ which were previously maximal in K.
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1. If σ is a maximal simplex in K:
   1.1 Remove or relocate all faces of σ which were previously maximal in K.
   1.2 Introduce the graph component corresponding to σ.
2. If σ is a critical simplex in K:
   2.1 Remove all faces of σ which were previously critical in K but are not any more.
   2.2 Introduce the graph component corresponding to σ.

$$\mathcal{O}\left(\Psi d_\sigma^2 \log \Psi\right)$$

Removal in CSD

Input: A simplex $\sigma = v_{\ell_0} \cdots v_{\ell_{d_\sigma}}$.

Task: Remove σ from K.

Removal in CSD

Input: A simplex $\sigma = v_{\ell_0} \cdots v_{\ell_{d_\sigma}}$.

Task: Remove σ from K.

1. Obtain the set Z_σ of critical simplices in K which contain σ.
2. For every $\tau \in Z_\sigma$, remove τ from K and insert the facets of τ which do not contain σ with appropriate filtration value.

$$\boxed{\mathcal{O}\left((\Psi d_\sigma + \Gamma_{d_\sigma}^2) d \log \Psi\right)}$$

Flag Complexes Construction

Flag Complex of G: All cliques of G are simplices of K.

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Output: A simplicial complex K with associated filtration.

1. Compute all maximal cliques in G. $\boxed{\mathcal{O}(k \cdot n^{\omega})}$
2. Sort all edges of G according to weight:

$$w(e_1) \geq w(e_2) \geq \cdots \geq w(e_{|E|}).$$

3. For every i from 1 to $|E|$:
   3.1 Find all maximal cliques in induced subgraph of $N(e_i)$.
   3.2 Remove e_i from G.

$\boxed{\mathcal{O}(\kappa \cdot n^{\omega})}$

Summary: ST vs. CSD

	Simplex Tree	Critical Simplex Diagram
Storage	$\mathcal{O}(m \log(nt))$	$\mathcal{O}(\kappa d \log(\kappa t))$
Filtration Query	$\mathcal{O}(d_\sigma \log n)$	$\mathcal{O}(d_\sigma \Psi \log \Psi)$
Insertion	$\mathcal{O}(2^{d_\sigma} d_\sigma \log n)$	$\mathcal{O}(d_\sigma^2 \Psi \log \Psi)$
Removal	$\mathcal{O}(m \log n)$	$\mathcal{O}((\Psi d_\sigma + \Gamma_{d_\sigma}^2) d \log \Psi)$
Construction of Flag Complex	$\mathcal{O}(md \log n)$	$\mathcal{O}(\kappa n^{2.38})$

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Thank you!

Value of Ψ

Data Set: Rips Complex from sampling of Klein bottle in $\mathbb{R}^5$.

No	t	$\lvert ST \rvert$	$\lvert CSD \rvert$	Γ_0	Γ_0^{avg}	Ψ	Ψ^{avg}
0	0	10,508,486	179,521	115	17.9	115	17.9
1	10	10,508,486	490,071	115	17.9	329	49.0
2	25	10,508,486	618,003	115	17.9	429	61.8
3	100	10,508,486	728,245	115	17.9	723	72.8
4	500	10,508,486	765,583	115	17.9	839	76.5
5	2,000	10,508,486	774,496	115	17.9	860	77.4
6	10,000	10,508,486	777,373	115	17.9	865	77.7
7	25,000	10,508,486	778,151	115	17.9	865	77.8
8	100,000	10,508,486	778,319	115	17.9	866	77.8
9	1,000,000	10,508,486	778,343	115	17.9	866	77.8
10	10,000,000	10,508,486	778,343	115	17.9	866	77.8