On the Sensitivity Conjecture for Disjunctive Normal Forms

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Joint work with Sébastien Tavenas (Université de Savoie-Mont-Blanc).

Boolean Function

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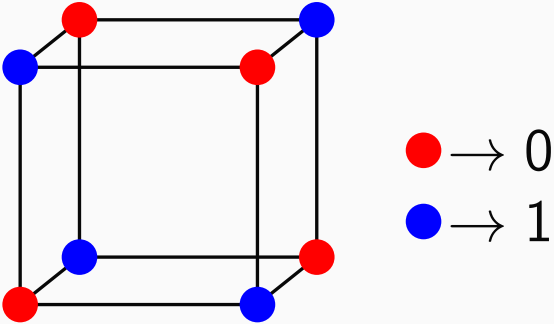
$$n = 3$$



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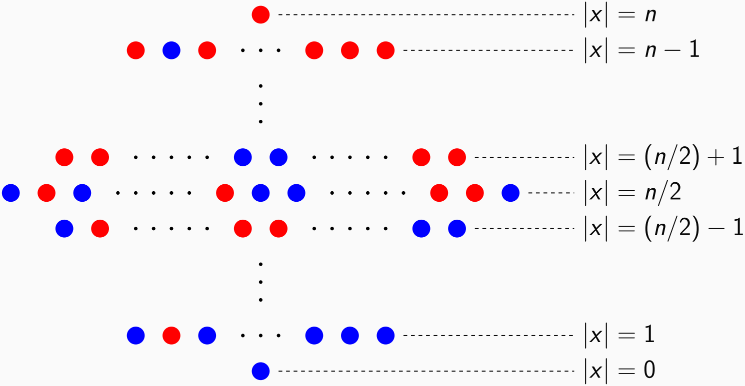

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$$s(f, x) = \sum_{i=1}^{n} f(x) \oplus f(x \oplus e_i)$$

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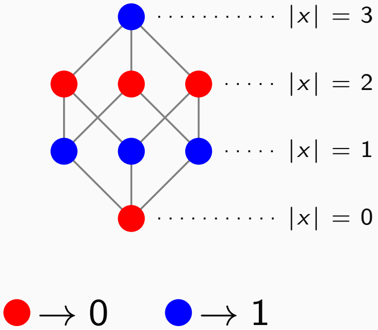

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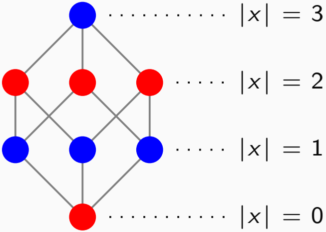

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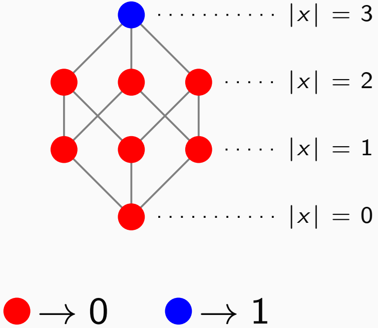

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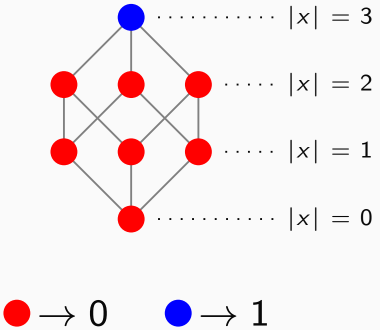

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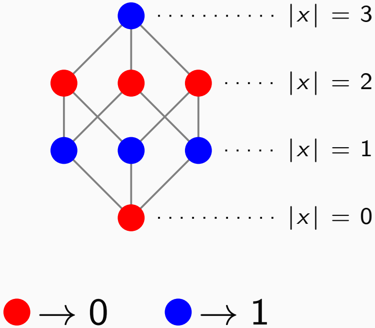

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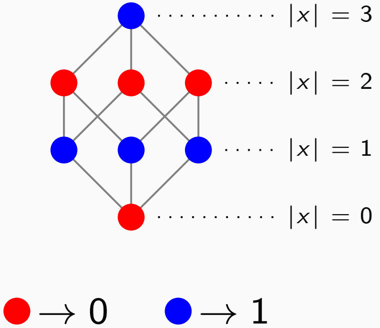

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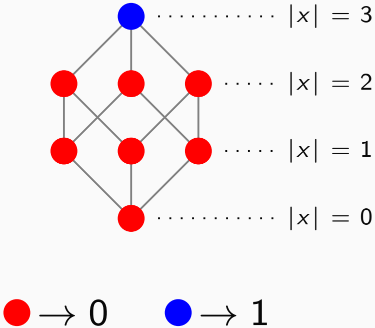

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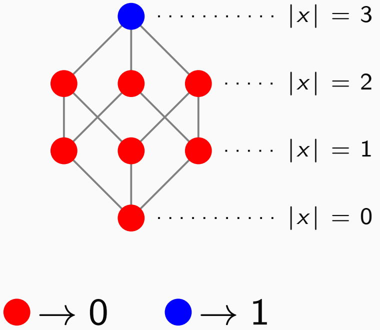

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$$\boxed{bs(f) \geq s(f)}$$

Sensitivity Conjecture: Statement

Conjecture (*Nisan and Szegedy, 1994*)

There exist constants c and δ such that for all Boolean functions we have:

$$bs \leq c \cdot s^{\delta}.$$

Sensitivity Conjecture: Equivalent forms

Theorem

The following statements are equivalent:

1. $bs \leq poly(s)$.
2. $deg, C, DT \leq poly(s)$.
3. *There exist constants* c *and* δ *such that for every induced subgraph* G *of* Q_n *with* $|V(G)| \neq 2^{n-1}$ *we have:*

$$\max\{\Delta(G), \Delta(Q_n - G)\} \geq c \cdot n^{\delta}.$$

Sensitivity Conjecture: Progress

Simon'83	$bs \leq 4^s s$
Kenyon & Kutin'04	$bs \leq \left(\frac{e}{\sqrt{2\pi}}\right) e^s \sqrt{s}$
Ambainis et. al'14	$bs \leq 2^{s-1}s - (s-1)$
Ambainis, Prusis, & Vihrovs'16	$bs \leq 2^{s-1}\left(s - \frac{1}{3}\right)$
He, Li, & Sun'16	$bs \leq \left(\frac{8}{9} + o(1)\right) 2^{s-1}s$

Sensitivity Conjecture: Known Separation

Rubinstein'95	$bs = \frac{1}{2}s^2$
Virza'11	$bs = \frac{1}{2}s^2 + \frac{1}{2}s$
Ambainis & Sun'11	$bs = \frac{2}{3}s^2 - \frac{1}{3}s$

Sensitivity Conjecture: Special Cases

Turán'84	Graph Property and Symmetric functions
Nisan'91	Monotone functions
Chakraborty'09	Min-term Transitive
Gao, Mao, Sun, & Zuo'12	Bipartite Graph Property
Bafna, Lokam, Tavenas, & Velingker'16	Constant depth Regular Read-k Formulas

Block Property

Wlog we assume $bs(f) = bs(f, 0^n)$ and $f(0^n) = 0$.

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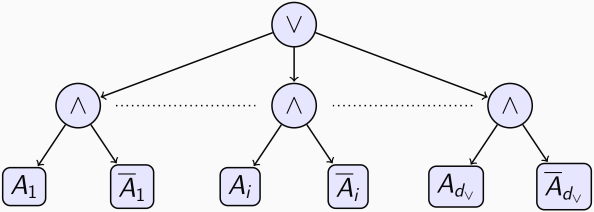

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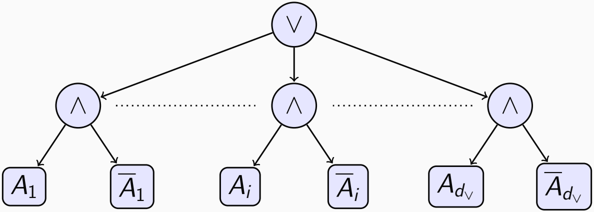

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Motivation: Captures best known separation examples.

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Theorem

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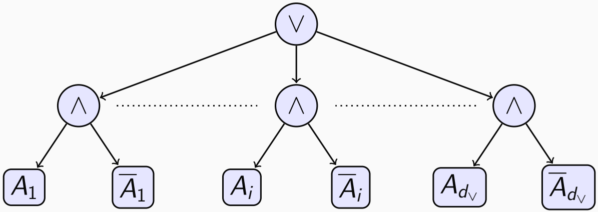
For every Boolean function f admitting the Block Property we have:

- $\star$ $bs(f) = d_\vee$.
- $\star$ $s(f) \geq d_\wedge/2$.
- $\star$ $s(f) \geq d_\vee/2d_\wedge$.

Block structure: $bs(f) = d_\vee$



- Note that A_is are pairwise disjoint.
- This implies $bs(f, 0^n, A = \{A_1, \ldots, A_{d_\vee}, A_{d_\vee+1}\}) \geq d_\vee$.

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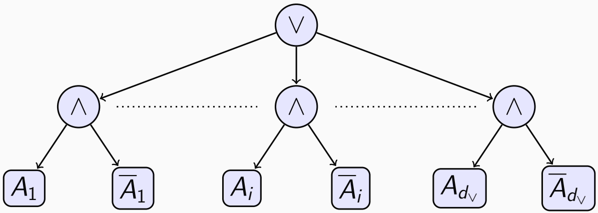

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- This implies $bs(f, 0^n, A = \{A_1, \ldots, A_{d_\vee}, A_{d_\vee+1}\}) \geq d_\vee$.
- $bs(f) \leq d_\vee$ follows from disjointness of blocks and assumption on $bs(f, 0^n)$.

Local argument: $s(f) \geq d_{\wedge}/2$

Fix $i^\star = \underset{i}{\text{argmax}}\ d_{\wedge_i}$. Let $x^\star = e_{A_{i^\star}}$.

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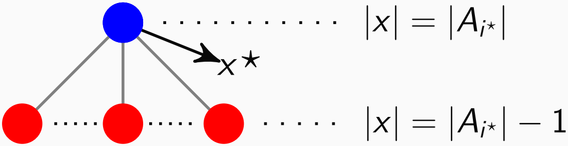

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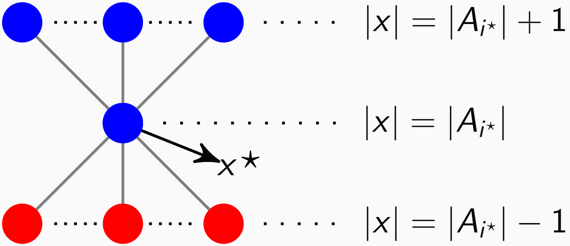

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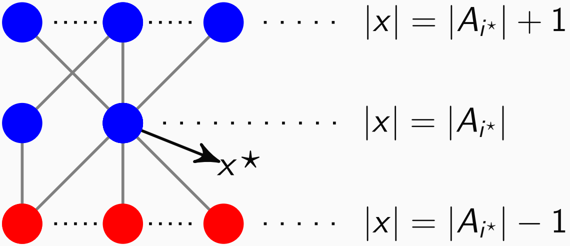

GLOBAL ARGUMENT: $s(f) \geq d_\vee / 2d_\wedge$

Consider the following graph G:

- $V(G)$: AND gates.
- $(\wedge_i, \wedge_j) \in E(G)$ iff $A_i \cap \overline{A}_j \neq \emptyset$ or $\overline{A}_i \cap A_j \neq \emptyset$.

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We have, $s(f, x) \geq d_\vee/2d_\wedge$.

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Theorem

For every Boolean function f admitting the Block Property we have:

$$bs(f) \leq 4s(f)^2.$$

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Theorem

Let f be a Boolean function admitting the t-block property, with $t \leq \frac{d_\vee}{d_\wedge^{1+\varepsilon}}$, for some $\varepsilon > 0$. Then, we have the following:

$$bs(f) \leq t\,(3s(f))^{1+\frac{1}{\varepsilon}}.$$

A Computational Question

Sensitivity Problem ($\mathcal{S}$): Given a circuit $C : \{0,1\}^n \to \{0,1\}$, $x \in \{0,1\}^n$, and blocks $B_1, \ldots, B_k$, the sensitivity problem is to find $y \in \{0,1\}^n$ such that:

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- $\mathcal{S}$ is in Search-NP.
- If (strong) Sensitivity conjecture is true then $\mathcal{S}$ is in TFNP.
- Is $\mathcal{S}$ in P?

End of the Talk

Thank you!