# Reversing Color Coding

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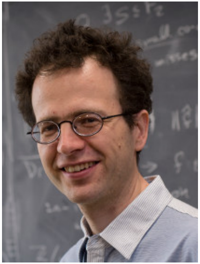
(Rutgers University)

Joint work with



Boris Bukh
(Carnegie Mellon University)

Bhargav Narayanan
(Rutgers University)

# Outline of Talk

- Colored vs. Uncolored Problems

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- Closest Pair Problem

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- Closest Pair Problem
- Parameterized Set Intersection Problem

# Colored versus Uncolored

# $k$-Clique

Uncolored $k$-Clique Problem:

**Input**: $G(V, E)$

**Output**: $k$-clique in $G$

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Input: $G(V_1 \dot\cup V_2 \dot\cup \cdots \dot\cup V_k, E)$

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Uncolored $k$-Clique Problem and Colored $k$-Clique Problem are computationally equivalent up to $O_k(1)$ factor

Uncolored $k$-Set Cover Problem:

Input: $S_1, \ldots, S_n \subseteq [n]$

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Uncolored and Colored $k$-Set Cover Problems are computationally equivalent up to $O_k(1)$ factor

Uncolored Clustering Problem:

Input: $P \subseteq \mathbb{R}^d$, $k \in \mathbb{N}$

Output: $P_1 \dot\cup P_2 \dot\cup \cdots \dot\cup P_k := P$ minimizing some clustering objective

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Is Clustering under Fairness constraints computationally harder than Standard Clustering?

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# Closest Pair

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Is Colored Closest Pair computationally harder than Uncolored Closest Pair?

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Can we reduce Colored version to Uncolored version?

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- BCP in $\ell_p$-metric when $d = 2^{O(\log^* n)}$ [W18, C18]

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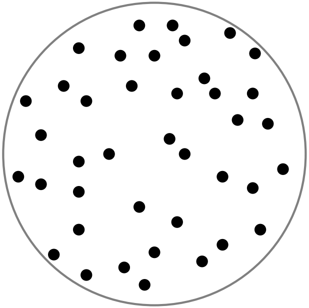

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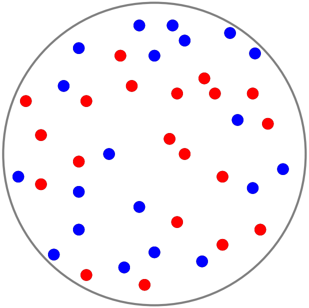

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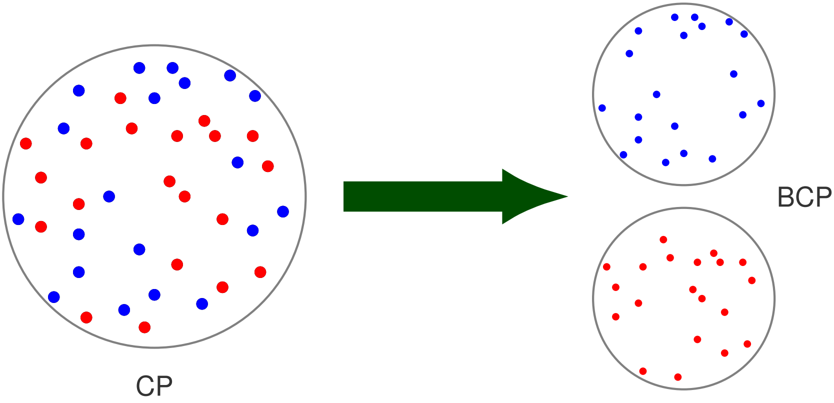

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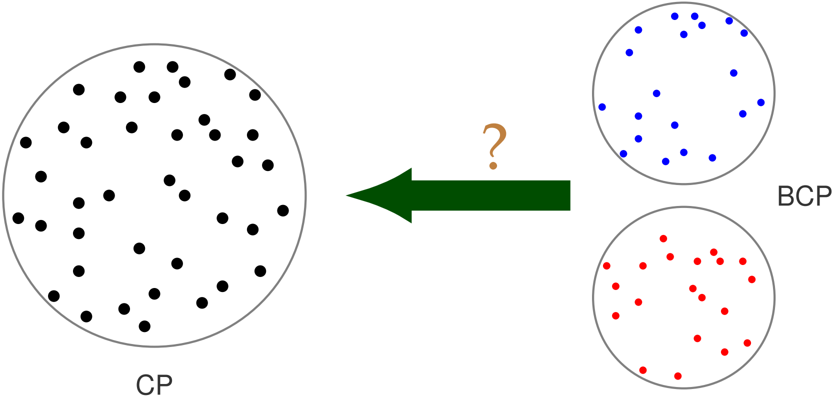

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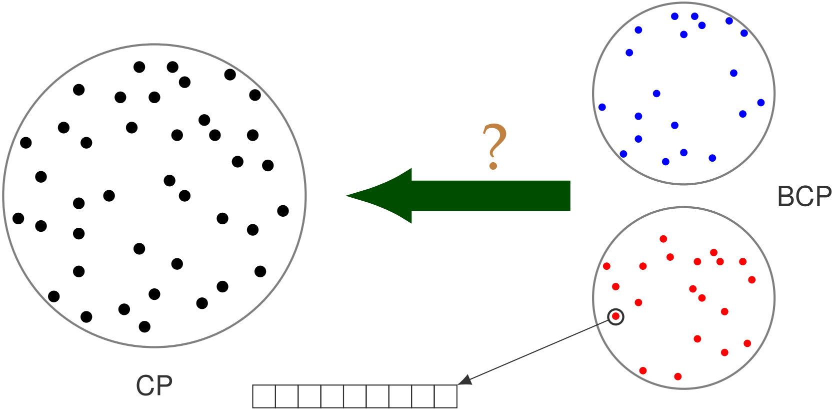

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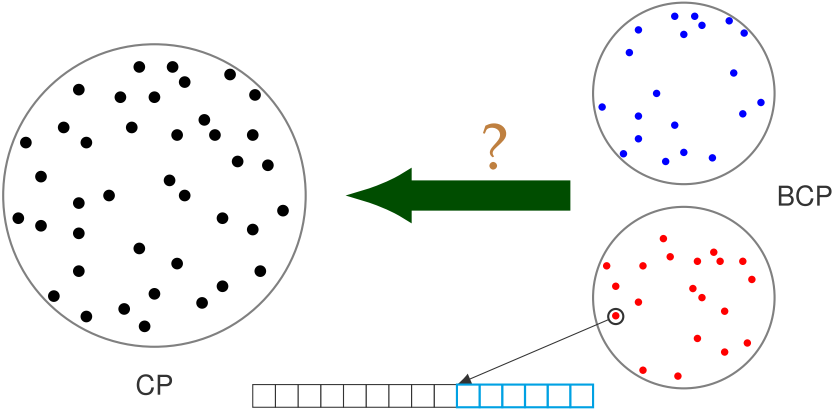

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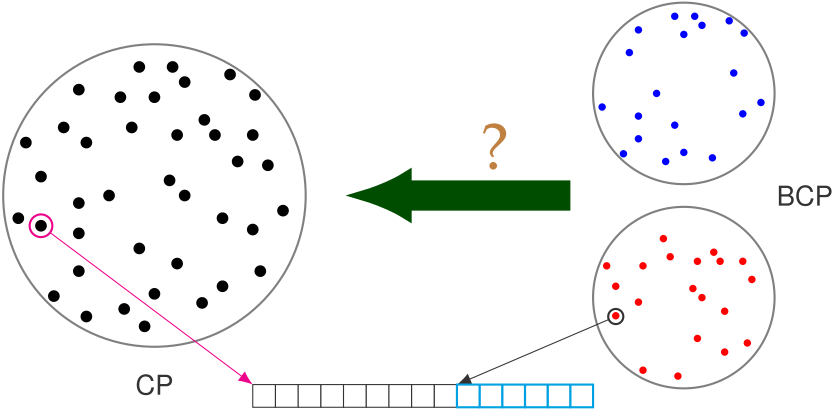

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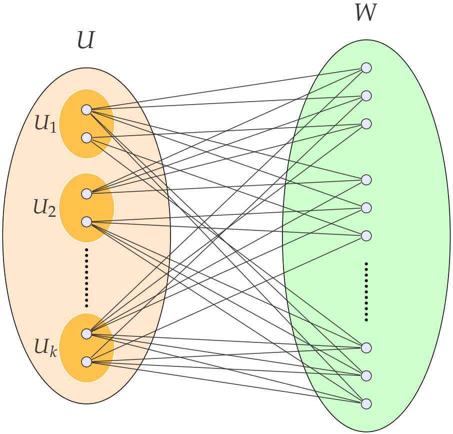
**Theorem (K-Manurangsi'18)**

- BCP and CP in $\ell_p$-metric are computationally equivalent when $d = (\log n)^{\Omega(1)}$.
- $(1+\delta)$-approximate BCP can be solved by $\tilde{O}(\sqrt{n})$ calls to $(1+\delta)$-approximate CP in $\ell_p$-metric when $d = \omega(\log n)$.

# Panchromatic Graphs



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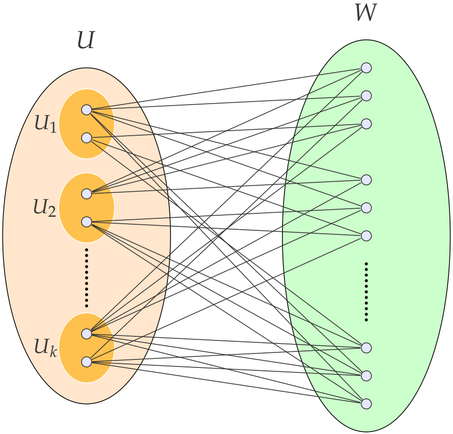


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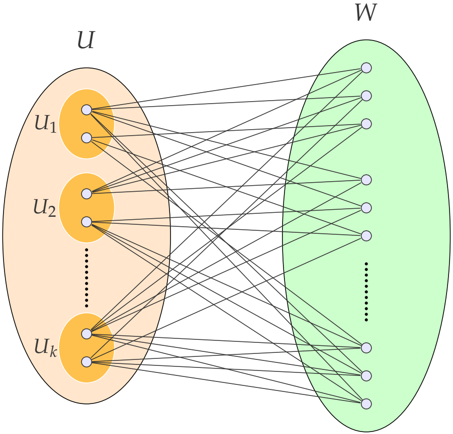


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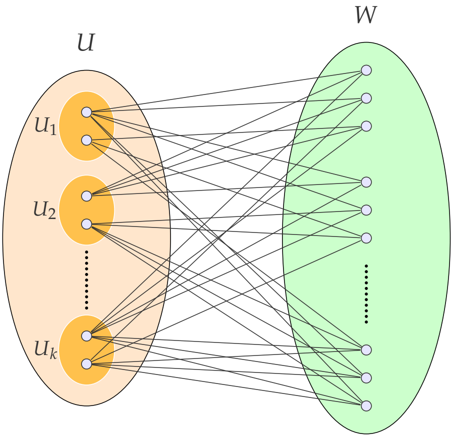


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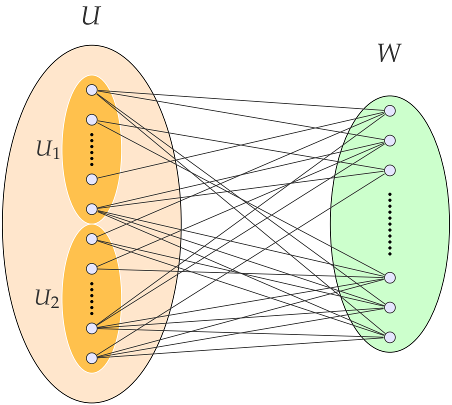

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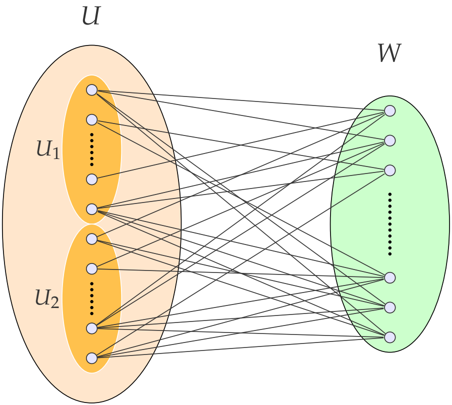


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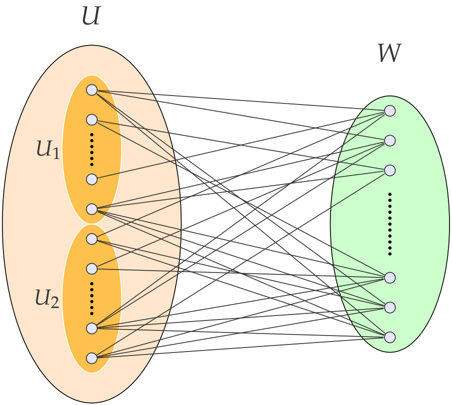


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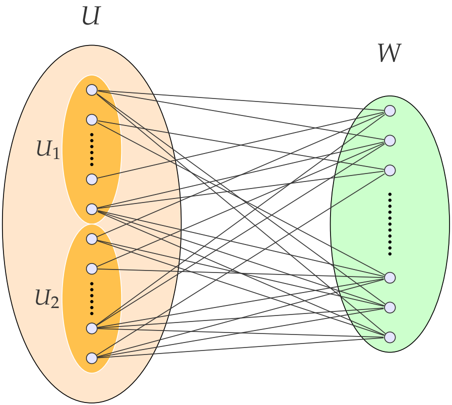


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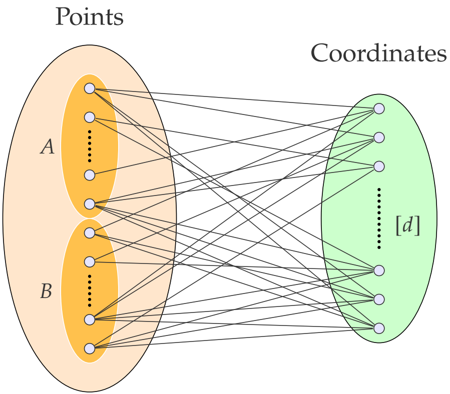

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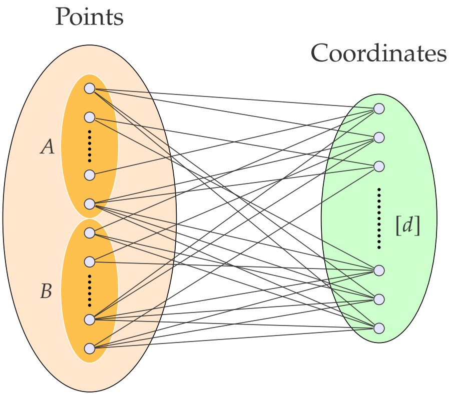


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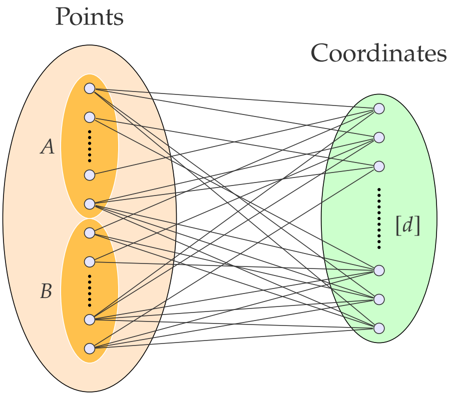


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Minimizing Distance
$\Updownarrow$
Maximizing Inner Product
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Maximizing Common Neighbors

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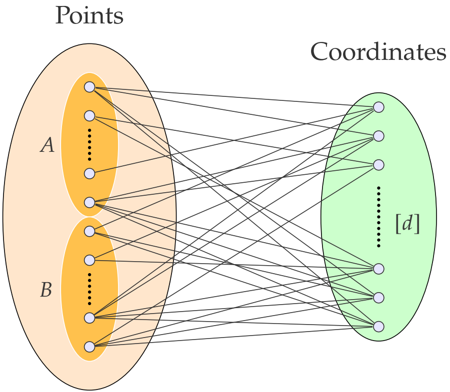

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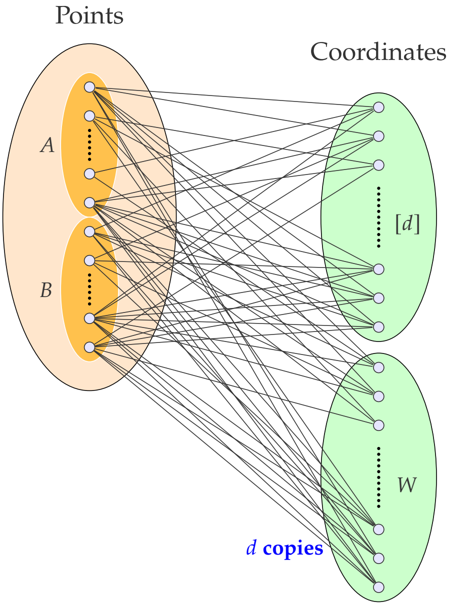
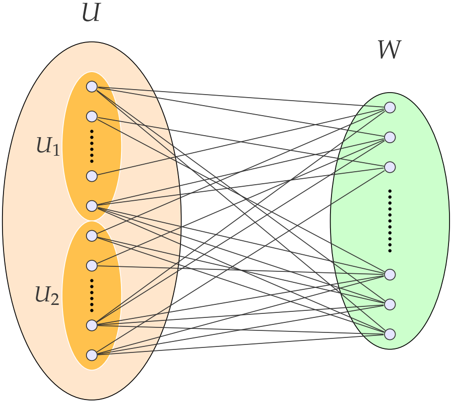


# Panchromatic Graphs when $k = 2$ [K-Manurangsi'18]




Many $(u_1, u_2)$ in $U_1 \times U_2$ has $t$ common neighbors in $W$

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# Construction of Panchromatic graphs when $k = 2$

*Polynomials are our friends.*

– TCS Folklore

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- $(p, (\alpha, \beta)) \in U \times W$ is an edge $\Leftrightarrow p(\alpha) = \beta$

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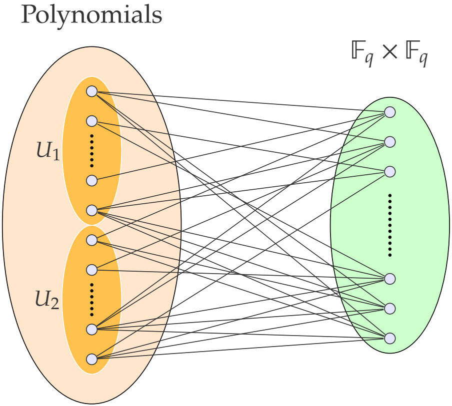

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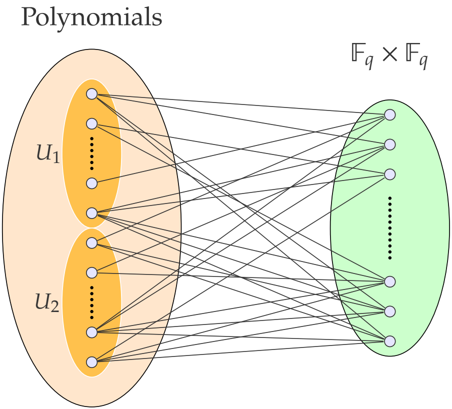


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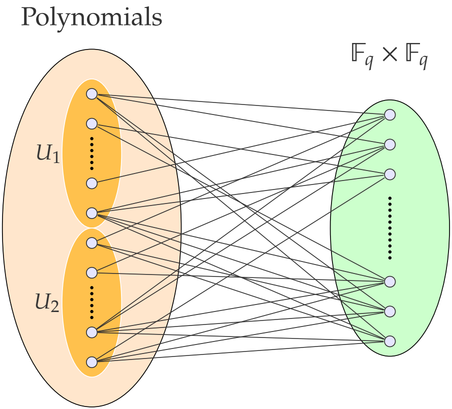


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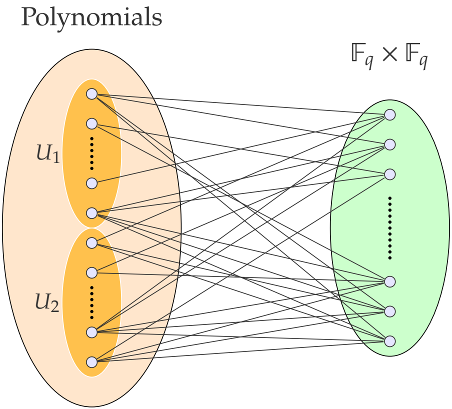


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They exist for $|W| = \text{polylog}(|U|)$!

**Theorem (K-Manurangsi'18)**

- BCP and CP in $\ell_p$-metric are computationally equivalent when $d = (\log n)^{\Omega(1)}$.
- $(1+\delta)$-approximate BCP can be solved by $\tilde{O}(\sqrt{n})$ calls to $(1+\delta)$-approximate CP in $\ell_p$-metric when $d = \omega(\log n)$.

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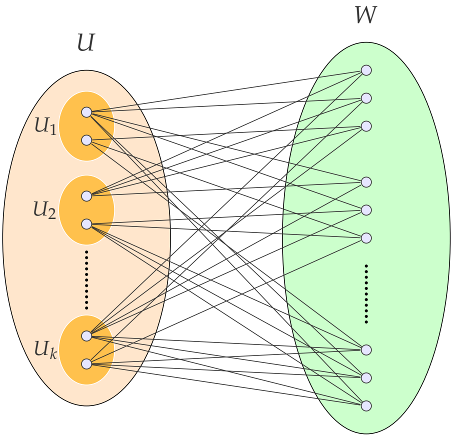


Many $(u_1, \ldots, u_k)$ in $U_1 \times \cdots U_k$ has $t$ common neighbors in $W$

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Do they exist for small $|W|$?

# Outline of Talk

- Colored vs. Uncolored Problems ✓
- Closest Pair Problem ✓
- Parameterized Set Intersection Problem

# Set Intersection

# $k$-Set Intersection

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**Output**: $S_{i_1}, \ldots, S_{i_k}$ whose intersection is maximized

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- Tight running time lower bounds under W[1]≠FPT, ETH, and SETH for exact version are straightforward!

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- $k$-Set Intersection and Colored $k$-Set Intersection are computationally equivalent up to $O_k(1)$ factors in run time.
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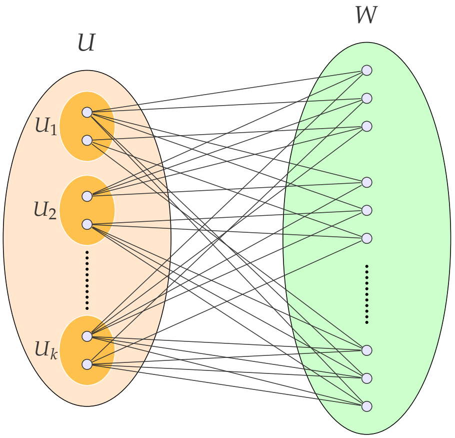
# Our Technical Result



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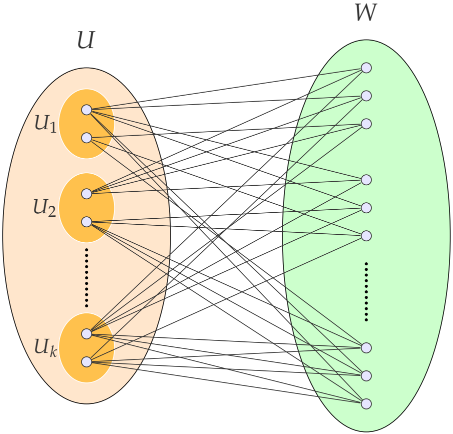


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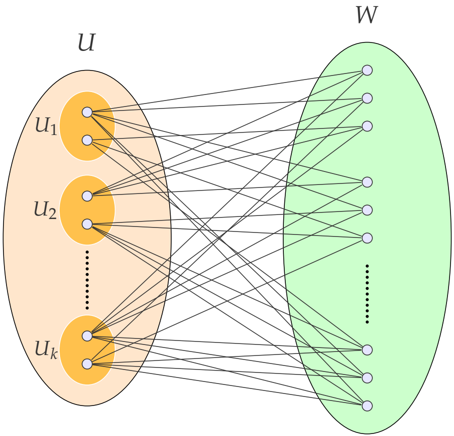


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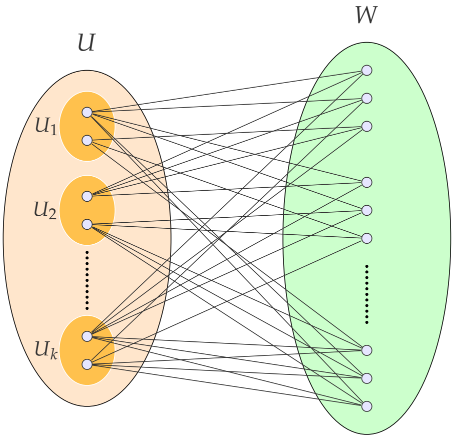


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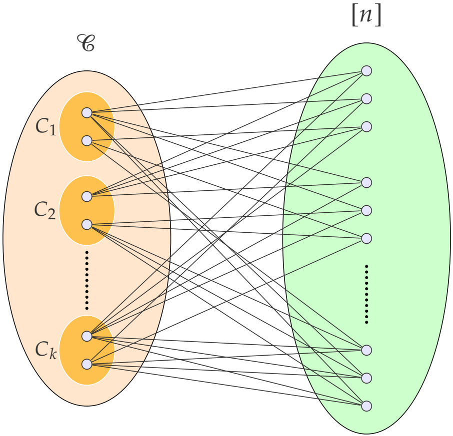
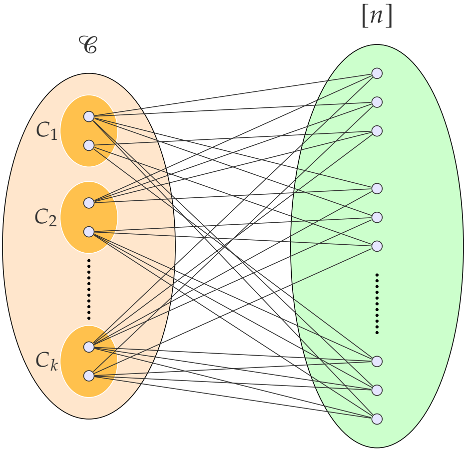
# Colored $k$-Set Intersection Problem




$$C_i = \{S_1^i, \ldots, S_n^i\}$$

# Panchromatic Graph Composition



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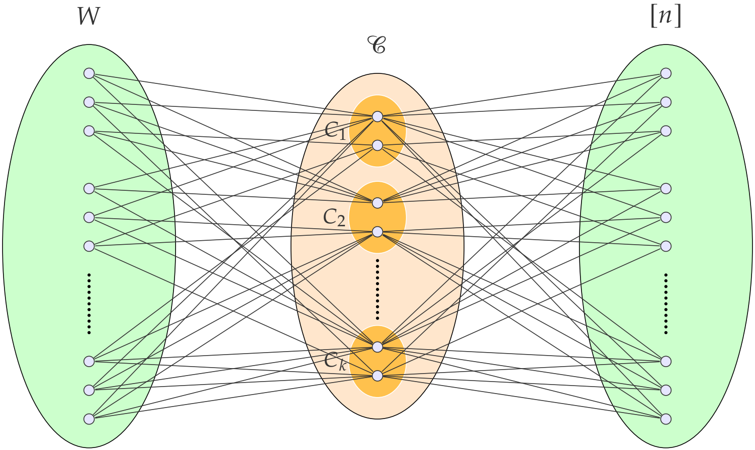

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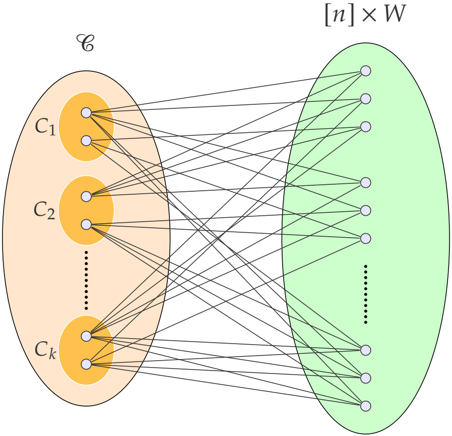


Edge between $S_i^j$ and $(a, w) \iff a \in S_i^j$ and edge between $S_i^j$ and $w$ in Panchromatic Graph

# Construction of Panchromatic graphs

*Polynomials are still our friends.*

– TCS Folklore

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**Theorem (Bukh-K-Narayanan'21)**

For $k, d \in \mathbb{N}$ and a prime power $q \in \mathbb{N}$, let $Z$ be the (random) number of common roots over $\mathbb{F}_q^k$ of $k$ independently chosen $k$-variate random $\mathbb{F}_q$-polynomials of degree $d$. Then, as $q \to \infty$, we have

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$$\Pr[Z > d^k] = O(q^{-d}).$$

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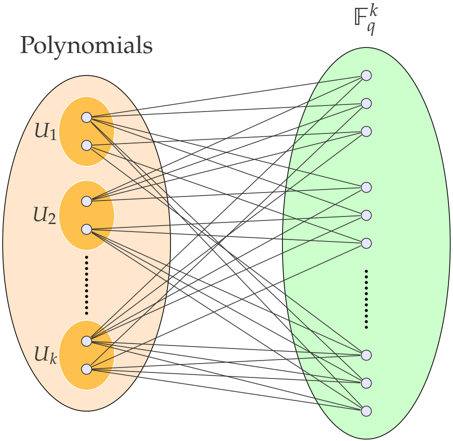


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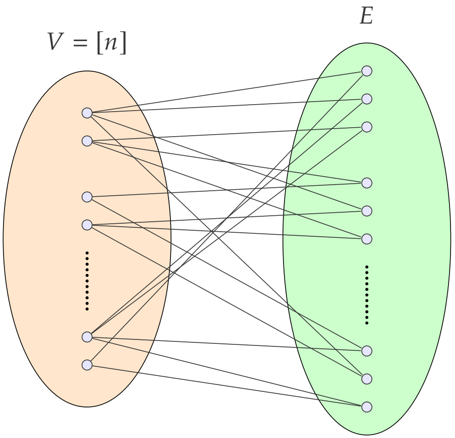
# Starting from $k$-Clique

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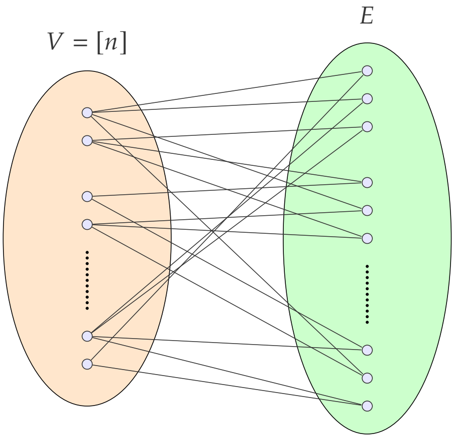


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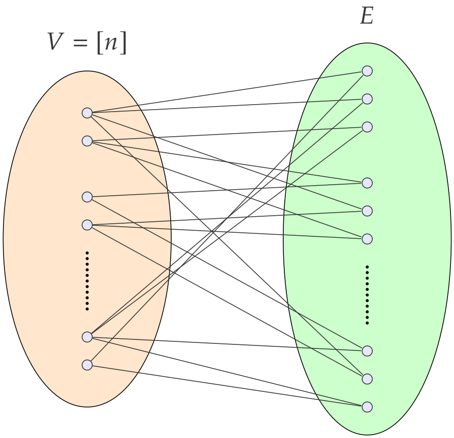


**Input**: $G([n], E)$

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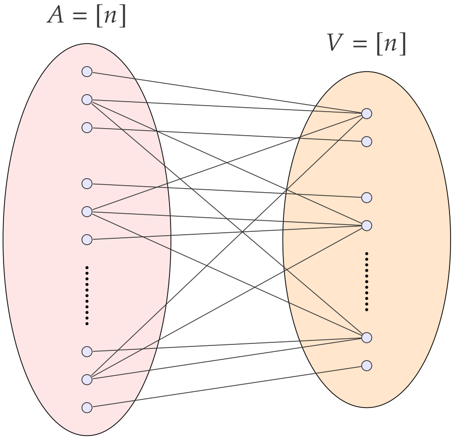
If $G$ has no $k$-clique then any $\binom{k}{2}$ vertices in $E$ has totally at least $k+1$ neighbors

# Threshold Graph



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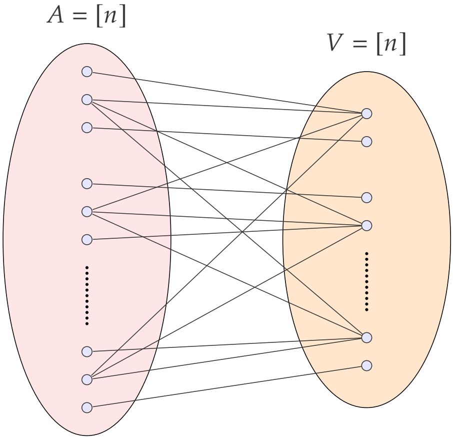


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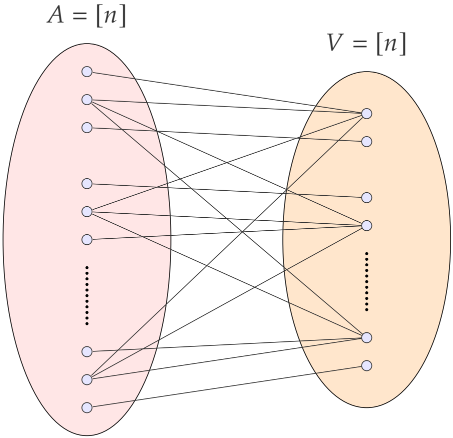


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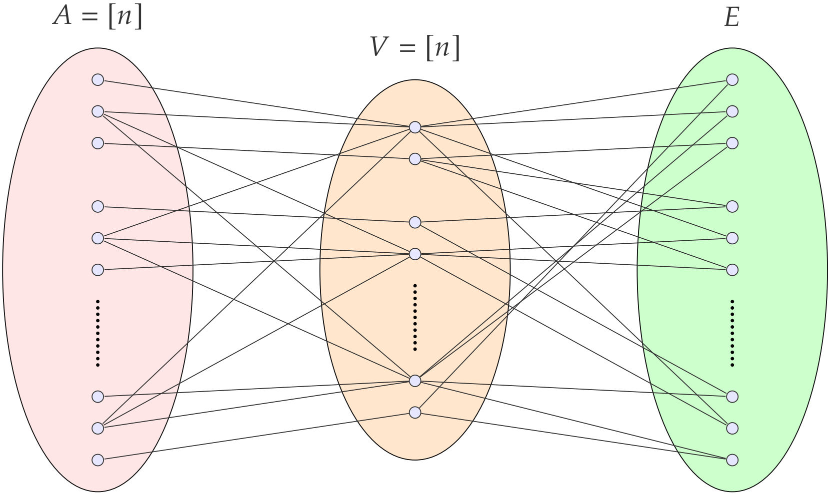

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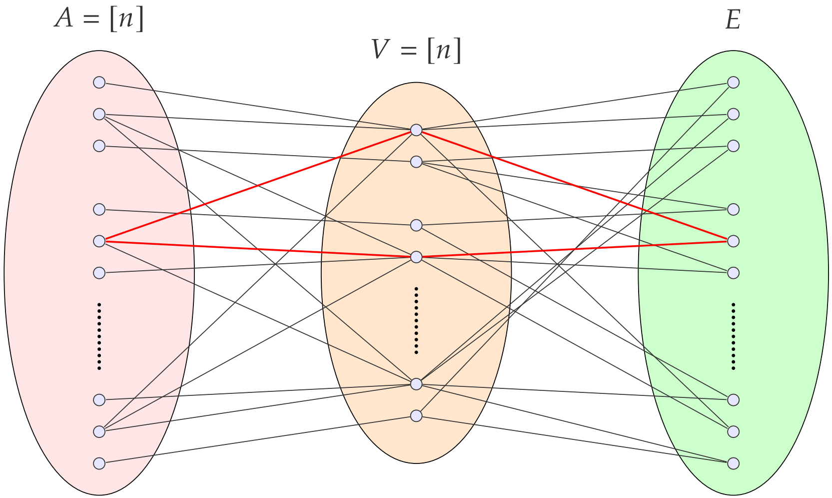


$(e, a) \in E \times A$ is an edge $\Leftrightarrow \exists\, v, v' \in V$ such that
$a$ and $e$ are common neighbors of $v$ and $v'$

# Completeness of Reduction

- Let $v_1, \ldots, v_k \in V$ be vertices of $k$-clique in $G$
- Let $A' \subseteq A$ be common neighbors of $v_1, \ldots, v_k$ in Threshold graph

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Completeness of Threshold Graph

Every $k$ vertices in $V$ has at least $n^{\Omega(1/k)}$ common neighbors in $A$

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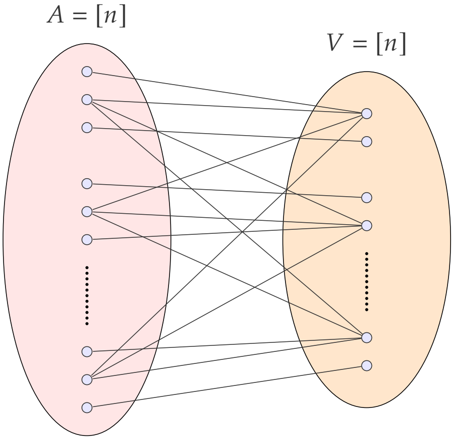
Soundness of Threshold Graph

Every $k + 1$ vertices in $V$ has at most
$k^{O(k)}$ common neighbors in $A$

# Threshold Graph

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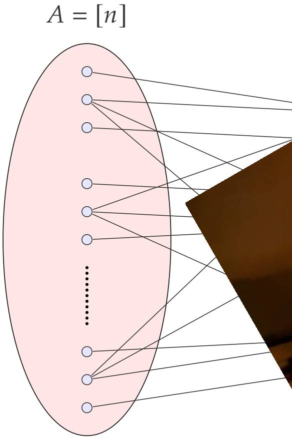


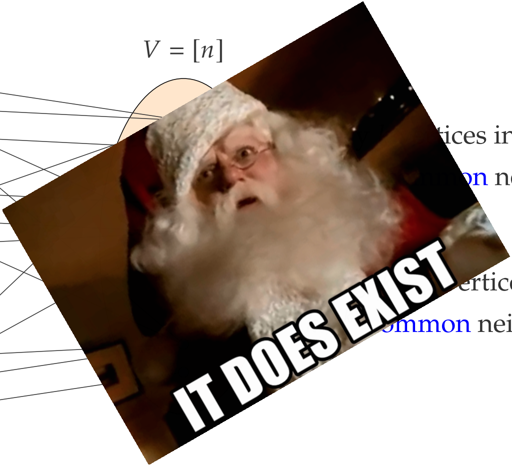


…tices in $V$ has at least …on neighbors in $A$

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# Outline of Talk

- Colored vs. Uncolored Problems ✓
- Closest Pair Problem ✓
- Parameterized Set Intersection Problem ✓

# Key Takeaways

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- Are there more applications for these graphs?

THANK
YOU!