Steiner Tree in ℓ_p-metrics

How hard is it to approximate?

Karthik C. S.
(Rutgers University)

Joint work with

Henry Fleischmann
(University of Michigan)

Surya Teja Gavva
(Rutgers University)

Wire routing in VLSI circuits

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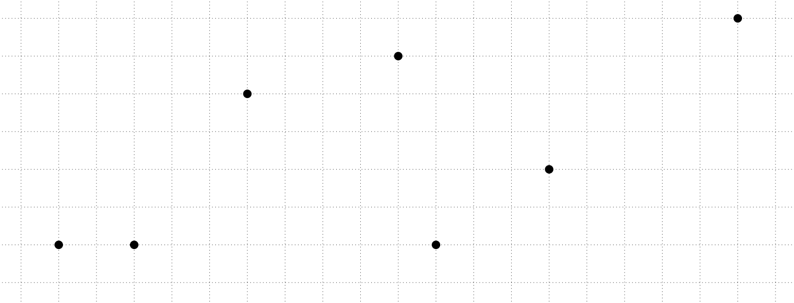

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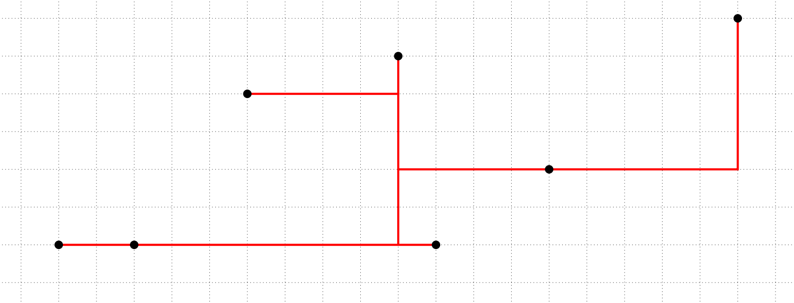

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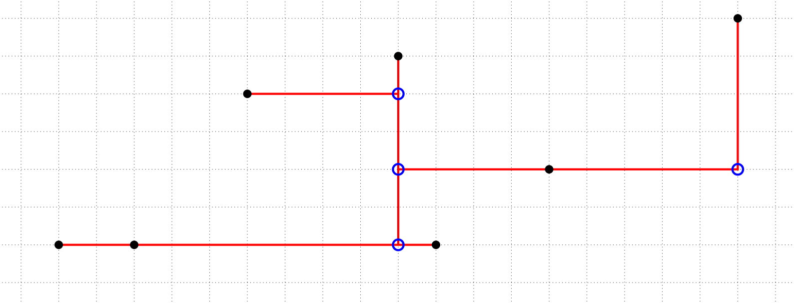
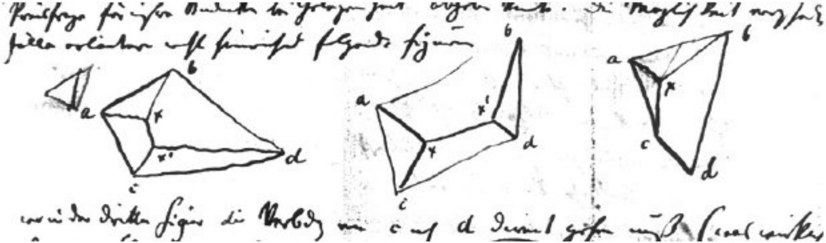
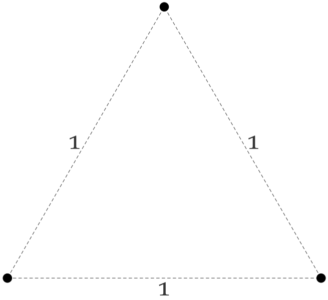


Gauss



Given four points in the Euclidean plane,
what is the cheapest network connecting them?

Steiner Tree in Euclidean Plane



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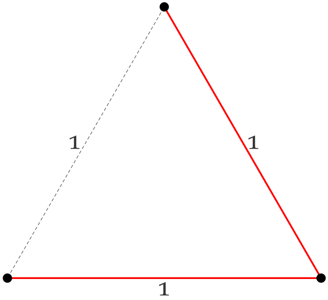

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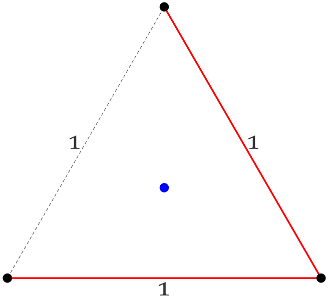

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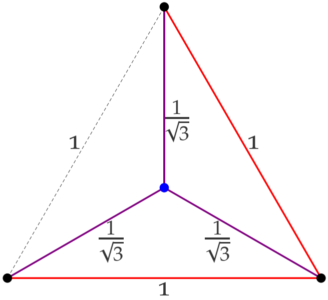

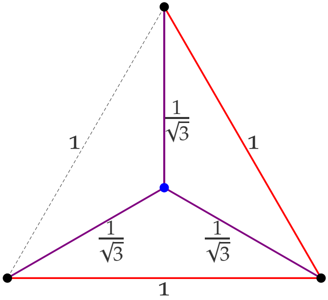

OPEN PROBLEM

Given n points in the Euclidean plane, show that the above configuration maximizes ratio of cost of Minimum Spanning Tree to cost of Minimum Steiner Tree

Quest for Computing Steiner Tree

So little we know and yet, we will continue to explore!

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Continuous Steiner Tree

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Discrete
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- CST is NP-hard in ℓ_0-metric (Foulds-Graham'82)
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Approximation Algorithms

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  - Compute Minimum Spanning Tree of only Terminals
- 1.39-approximation for DST in General metrics (Byrka-Grandoni-Rothvoß-Sanitá'10)
- PTAS for CST in fixed dimensional ℓ_2 metric (Arora'96)
  - PTAS for CST in fixed dimensional ℓ_p metrics

Hardness of Approximation

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Hardness of Approximation: Questions

Euclidean metric

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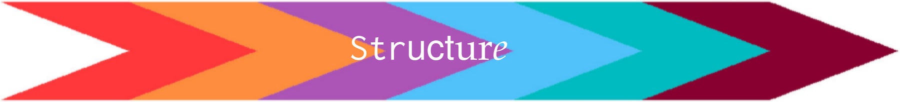
Can we rule out PTAS for DST in other high dimensional ℓ_p-metrics?

Hardness of Approximation: Questions

My Project

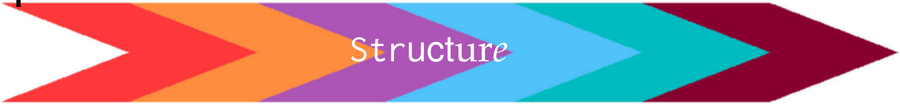
Spectrum of Computational Problems



Spectrum of Computational Problems

Circuit-SAT

CSP



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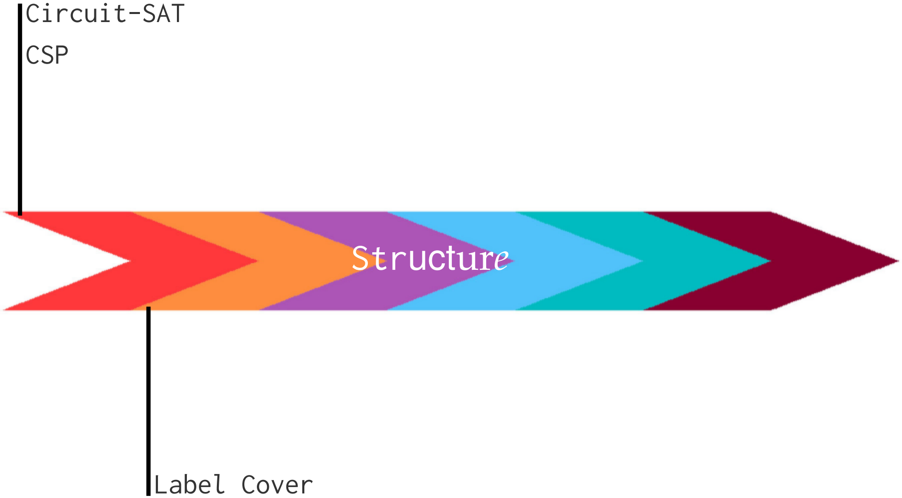

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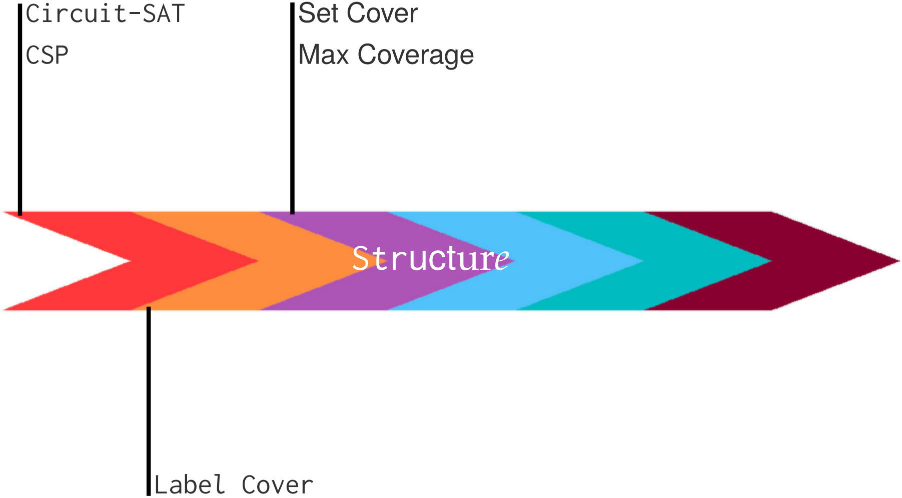

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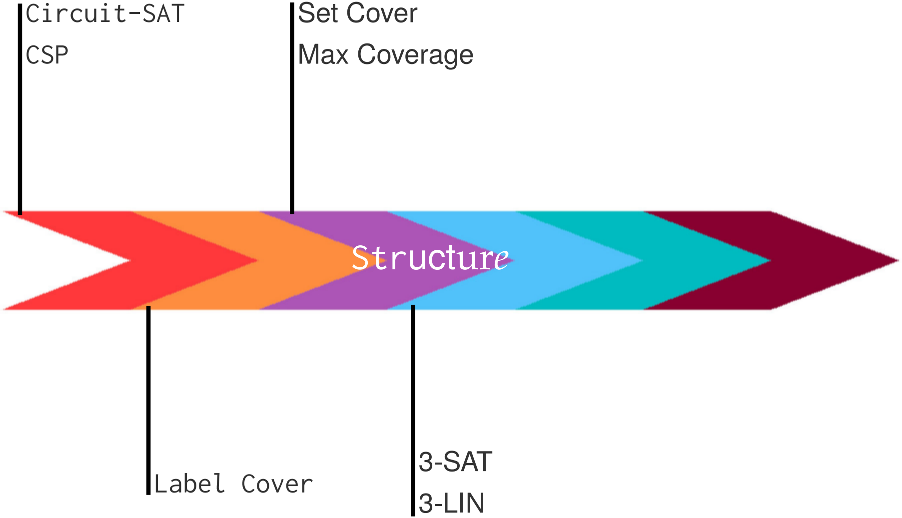

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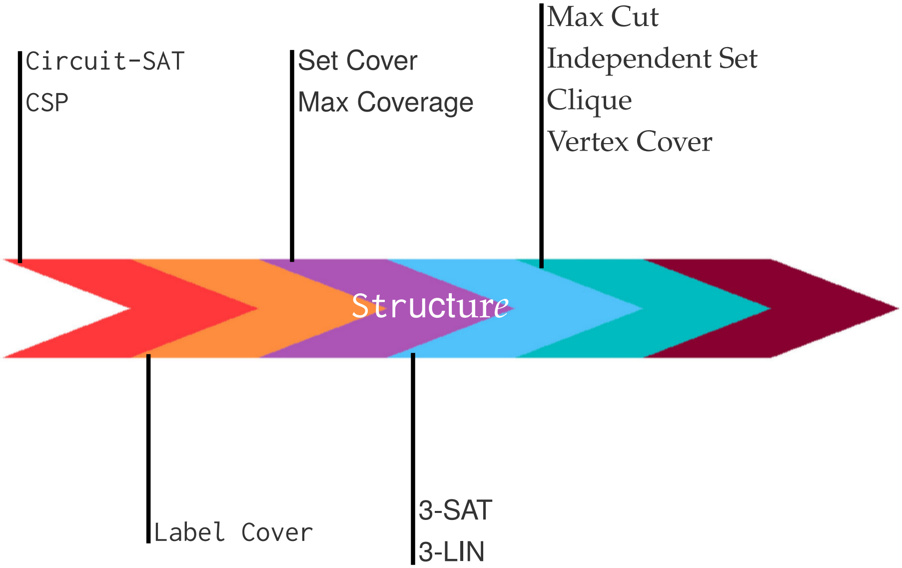

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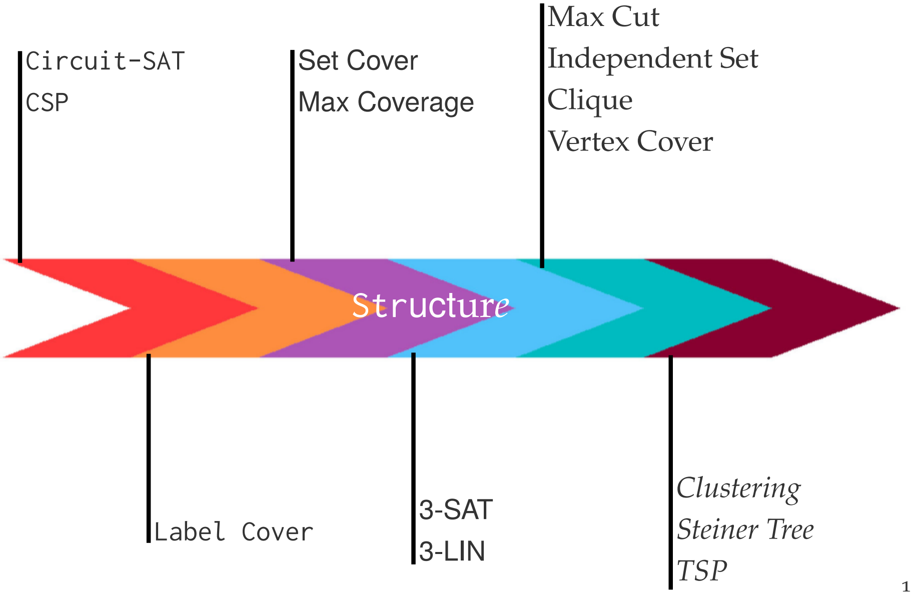

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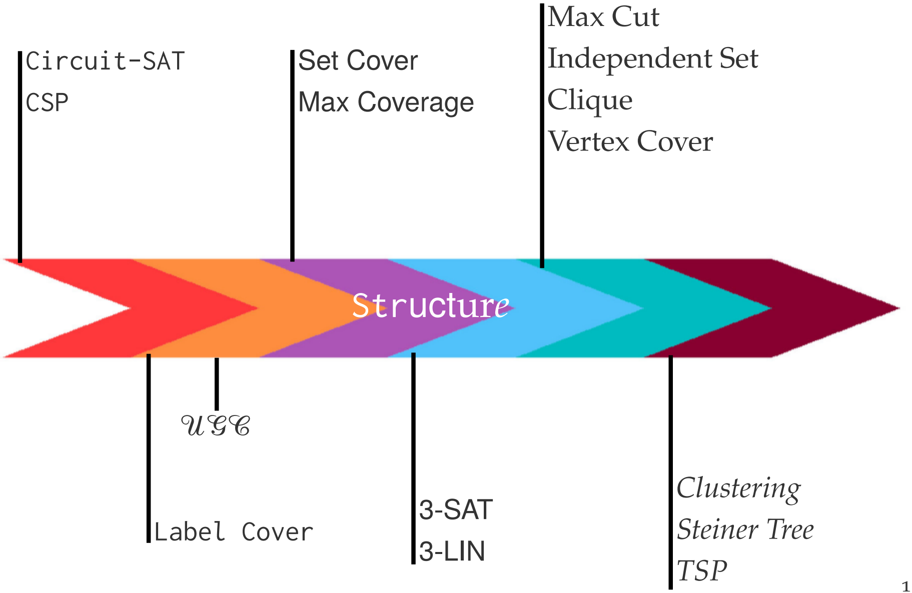

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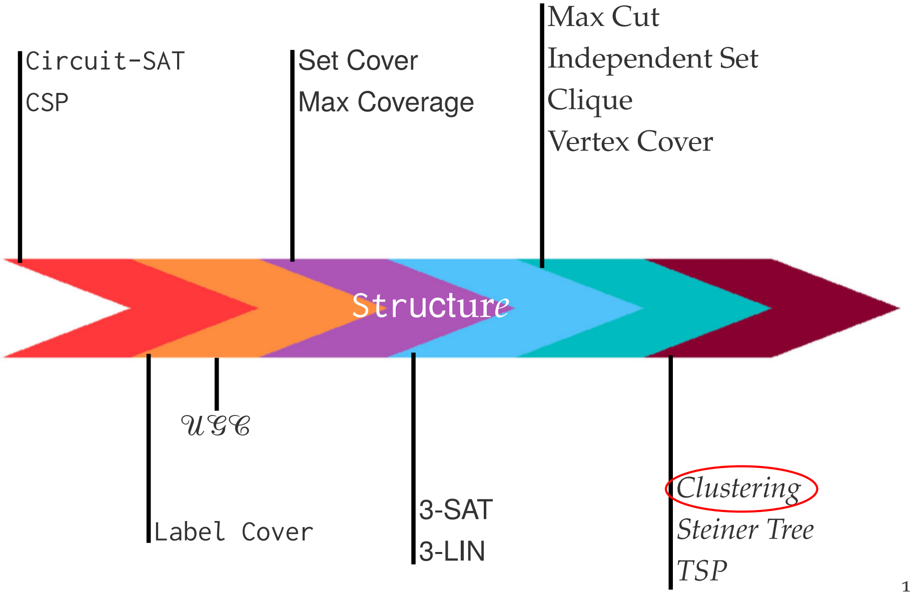

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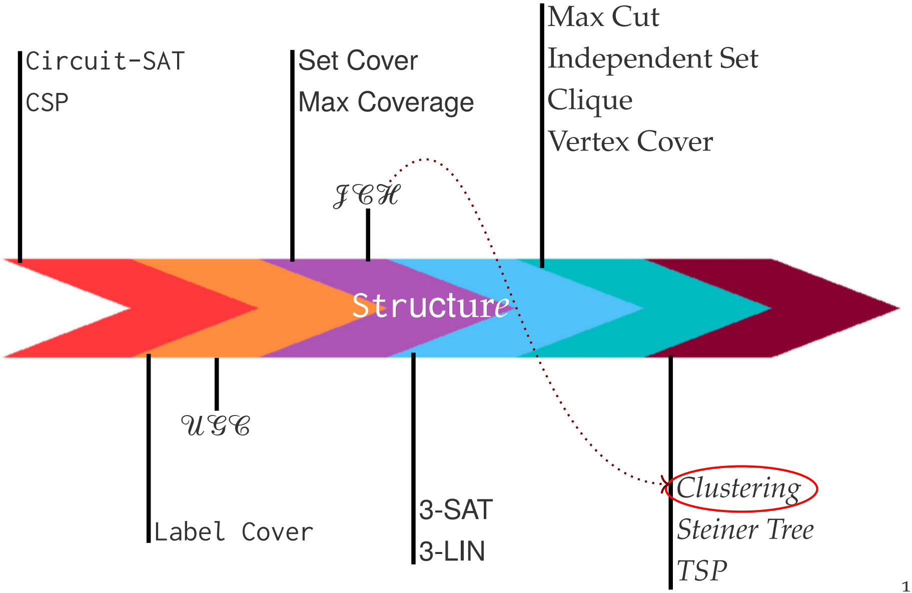

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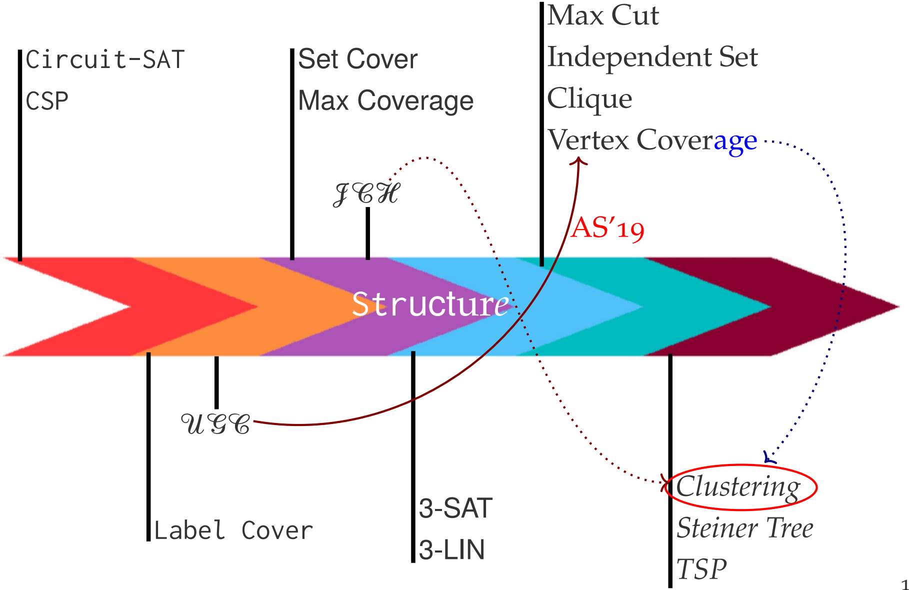

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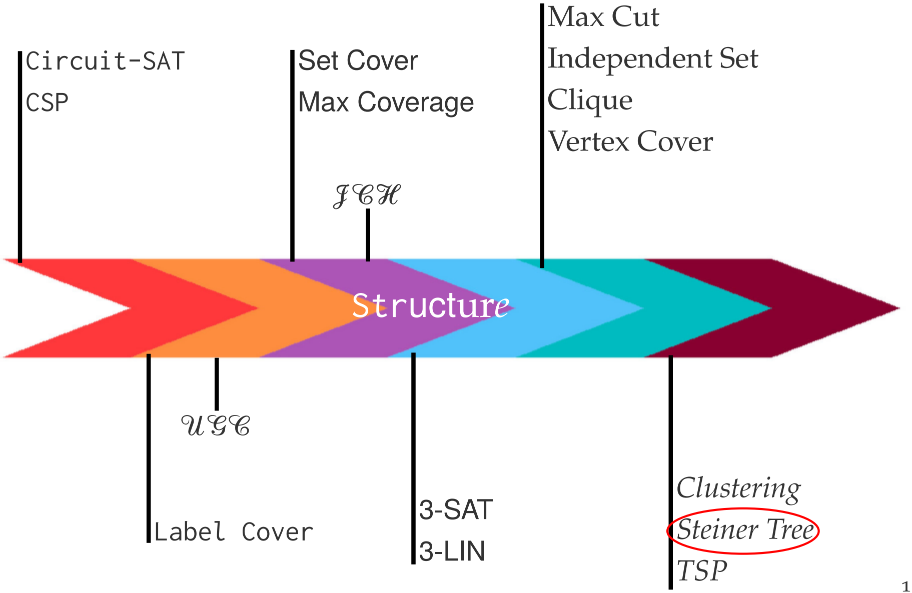

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- Above result holds even in $O(\log n)$ dimensions
- No PTAS for DST in Euclidean metric
  - Proof gives new insights into the difficulty of proving hardness for Euclidean Steiner Tree

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- Key ingredient: Steiner Tree decomposition through Terminal-Terminal edges (Bartal-Gottlieb'21)

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Theorem (Fleischmann–Gavva–K'23)

There is a poly time reduction from a graph G on n vertices to an instance of CST in the ℓ_∞-metric such that the optimal cost of the Steiner tree is $(n + \chi(G))/2$.

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Vertex Cover:

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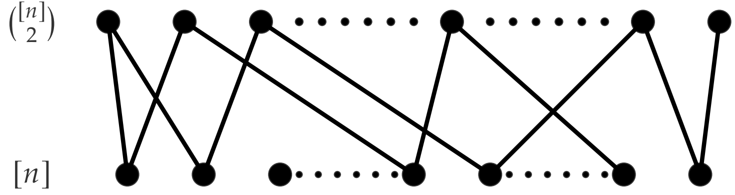
Inapproximability of DST in Hamming metric



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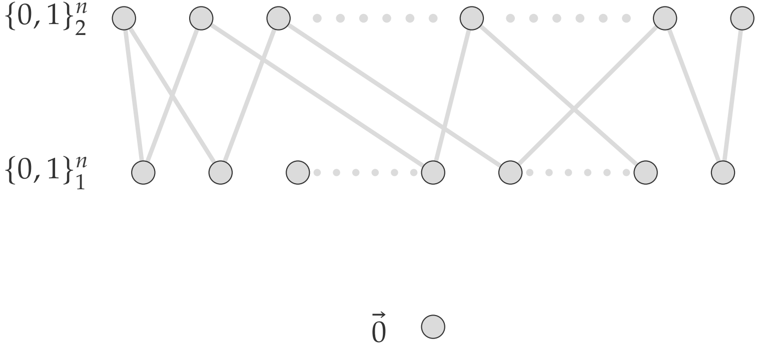

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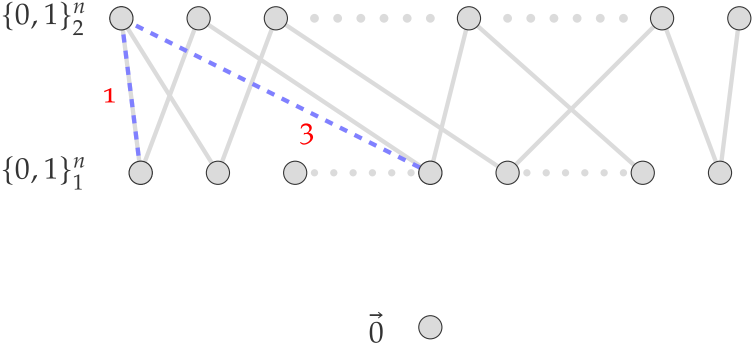

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Vertex Cover to Euclidean Steiner Tree

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All these obstacles are for DST
The obstacles for CST are way more serious!

3-Set Packing:

- Input: Set System $(U, \mathscr{C})$, $\mathscr{C} \subseteq \binom{[n]}{3}$
- Objective: Maximum size subcollection of $\mathscr{C}$ which are pairwise disjoint

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Theorem (Petrank'94)

For some $\varepsilon > 0$, it is NP-hard to distinguish:

YES: There are $n/3$ pairwise disjoint subsets in the input

NO: There are at most $(1-\varepsilon) \cdot n/3$ pairwise disjoint subsets in the input

3-Set Packing to Euclidean DST

- **Terminals:** Universe $\longrightarrow$ $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$
- **Facilities:** Sets $\longrightarrow$ $\{\lambda \cdot \vec{e}_i + \lambda \cdot \vec{e}_j + \lambda \cdot \vec{e}_k : \{i, j, k\} \in \mathscr{C}\}$

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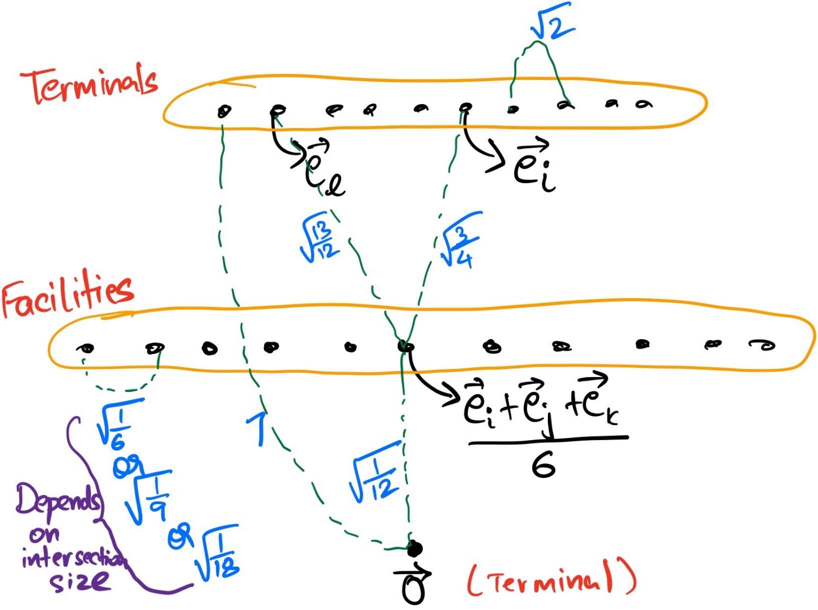
Structural Claim

Steiner points used form the maximum packing of sets

Structural Picture



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- Steiner Tree cost is:

$$(n/3) \cdot \sqrt{\frac{1}{12}} + n \cdot \sqrt{\frac{3}{4}}$$

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$$\text{Cost of } T = (n/3)(1-\varepsilon)\sqrt{\tfrac{1}{12}} + n(1-\varepsilon)\sqrt{\tfrac{3}{4}} + \varepsilon n \cdot 1$$

(ε, δ)-3-Set Packing

(ε, δ)**-3-Set Packing**:

- Input: Set System $(U, \mathscr{C})$, $\mathscr{C} \subseteq \binom{[n]}{3}$
- Completeness: There are $n/3$ pairwise disjoint subsets in $\mathscr{C}$
- Soundness: There are at most $(1-\varepsilon)n/3$ pairwise disjoint subsets in $\mathscr{C}$ and every set cover must be of size at least $(1+\delta)n/3$

Our result on DST

Theorem (Fleischmann–Gavva–K'23)

Assuming (ε, δ)-3-Set Packing is NP-hard, we have that DST in ℓ_p-metric is NP-hard to approximate to $(1+\gamma)$ factor, where

$$\gamma := \begin{cases} \delta/4 & \text{if } p = \infty \\ \frac{\varepsilon}{2}\left(1 - \frac{1}{3^{1/p}}\right) + 2\delta\left(\frac{1}{2\cdot 3^{1/p}} - \frac{3}{8}\right) & \text{if } p > 1/\log_3(4/3) \\ \varepsilon/8 & \text{if } p = 1/\log_3(4/3) \approx 3.8 \\ \varepsilon/26 & \text{if } p = 2 \\ > 0 & \text{if } p \in (1,2) \cup \left(2, \frac{1}{\log_3(4/3)}\right) \end{cases}$$

Proof Sketch of inapproximability of CST in ℓ_∞-metric

Theorem (Fleischmann–Gavva–K'23)

There is a poly time reduction from a graph G on n vertices to an instance of CST in the ℓ_∞-metric such that the optimal cost of the Steiner tree is $(n + \chi(G))/2$.

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- Cost of Tree $= 0.5 \cdot n + 0.5 \cdot \chi(G)$

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Key Takeaways

- Ruling out PTAS for Euclidean CST is still open!
- No PTAS for DST in ℓ_p-metrics
- No PTAS for CST in ℓ_∞-metric
- DST is at least as hard as CST
- At the heart of Steiner Tree Computation lies:
  - 3-Set Cover
  - 3-Set Packing
  - $n/3$-Chromatic number

THANK
YOU!