On Equivalence of Parameterized Inapproximability of k-median, k-max-coverage, and 2-CSP

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Joint Work with

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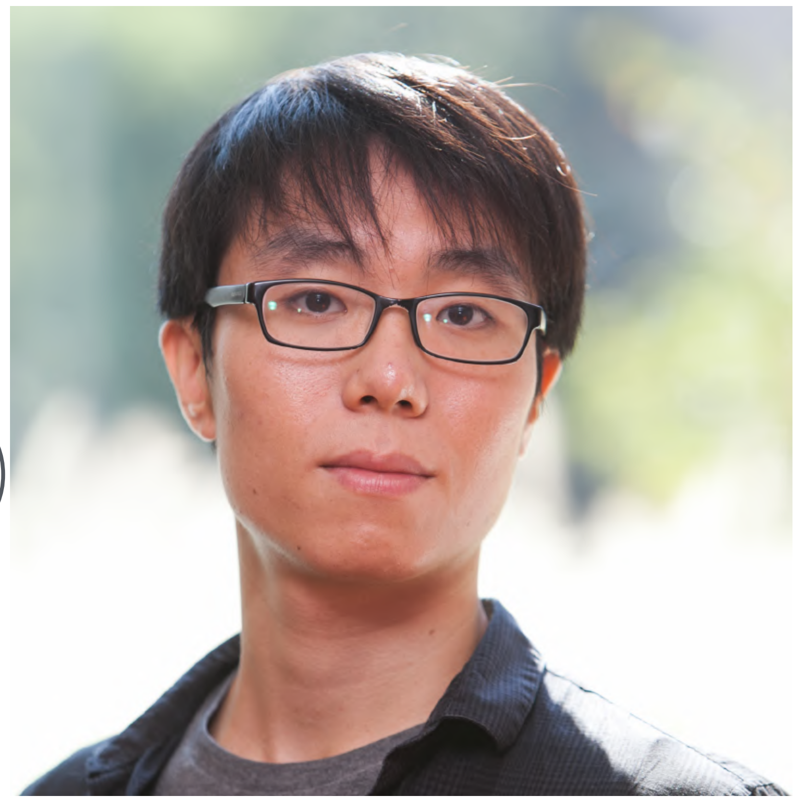

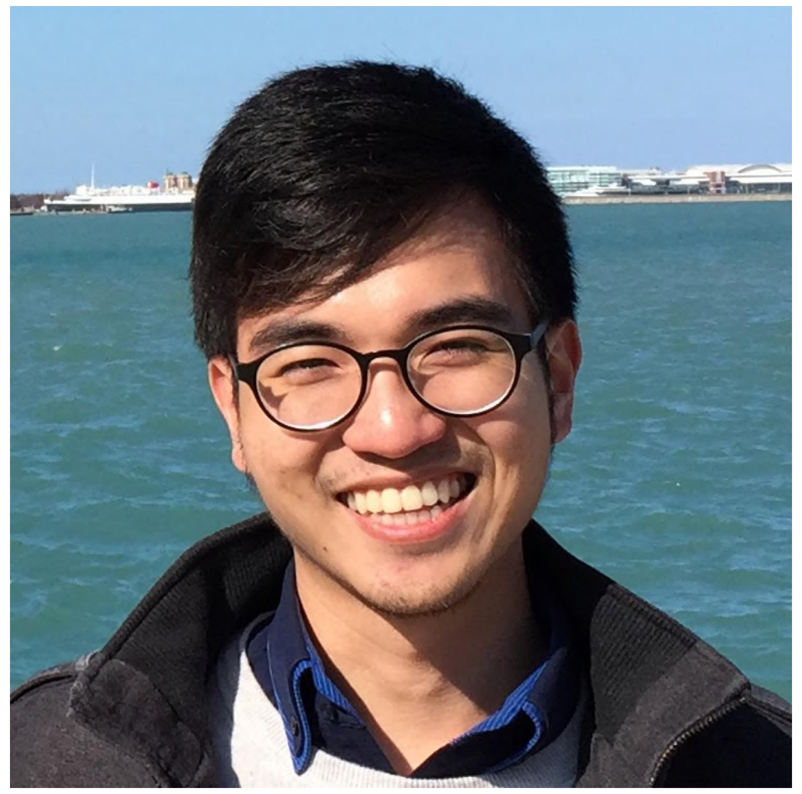

Pasin Manurangsi
(Google Thailand)

2-CSP k-max-coverage k-median

2-CSP   k-max-coverage   k-median

- **Input**: Graph $G = (V, E)$ and constraints:

$$\mathscr{C} = \{C_e \subseteq \Sigma \times \Sigma\}_{e \in E}$$

- **Output**: Assignment $\sigma : V \to \Sigma$ maximizing:

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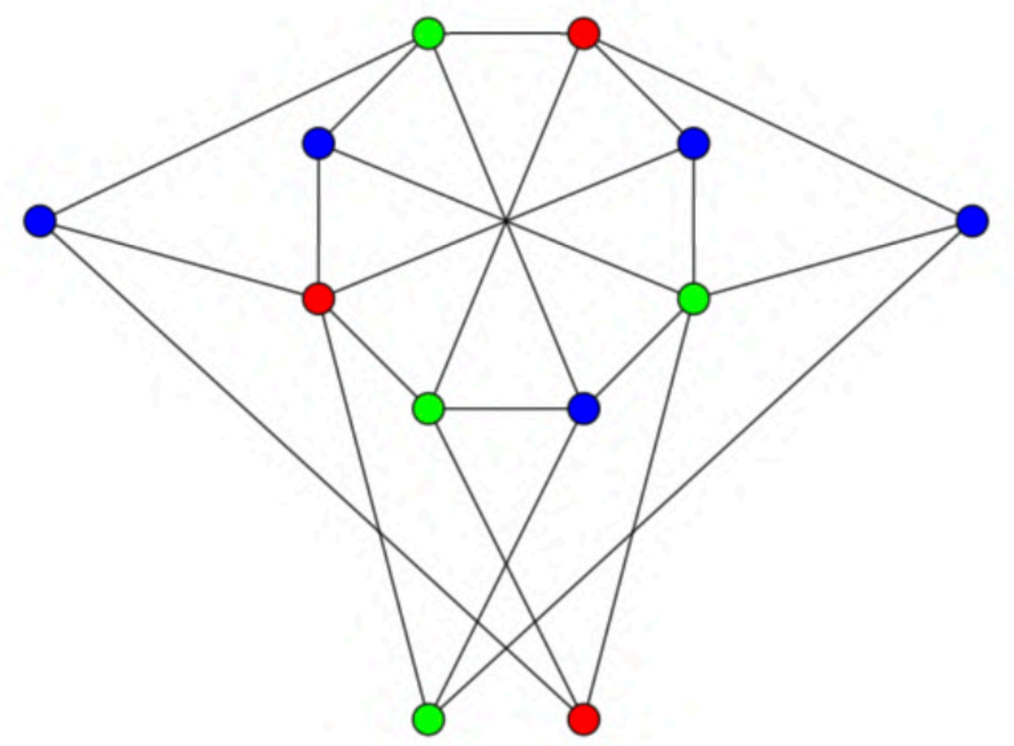

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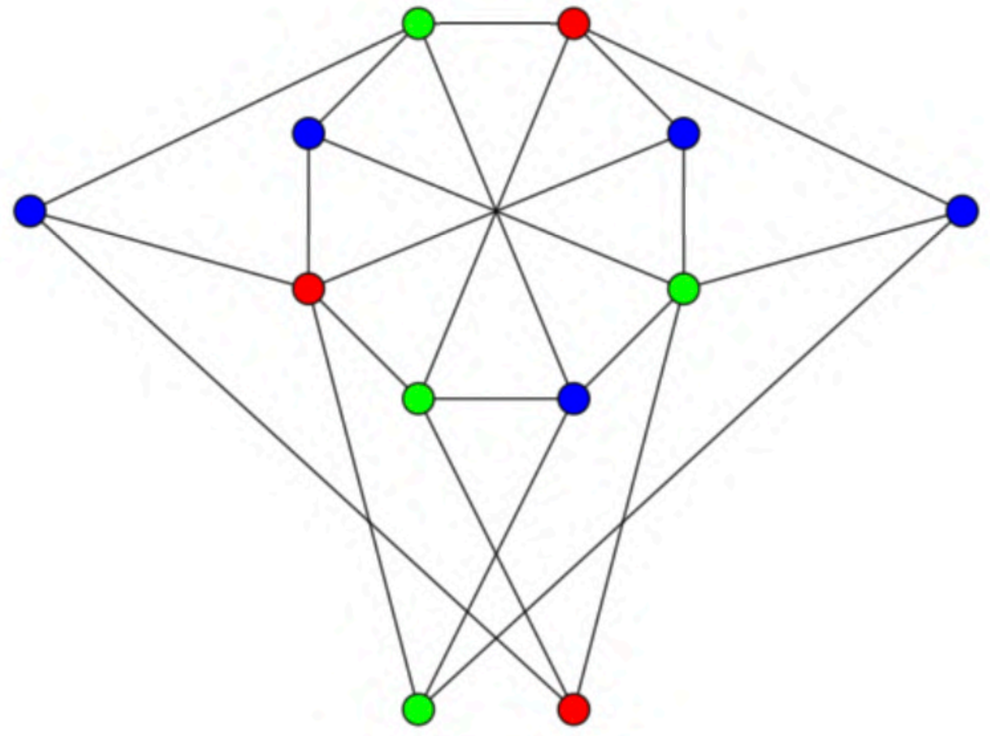

k-max-coverage

- **Input**: $S_1, \ldots, S_m \subseteq [n]$, and integer k
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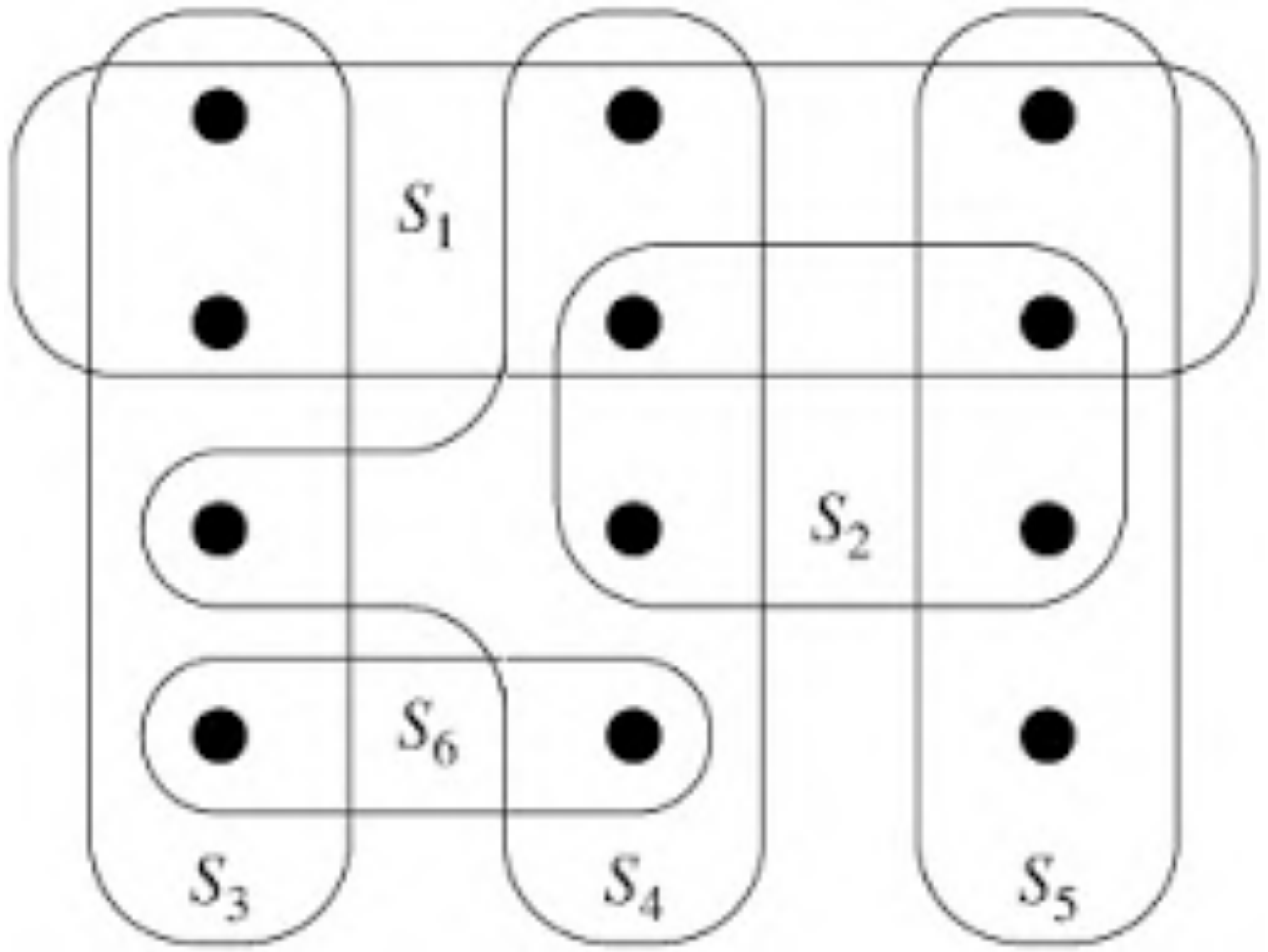

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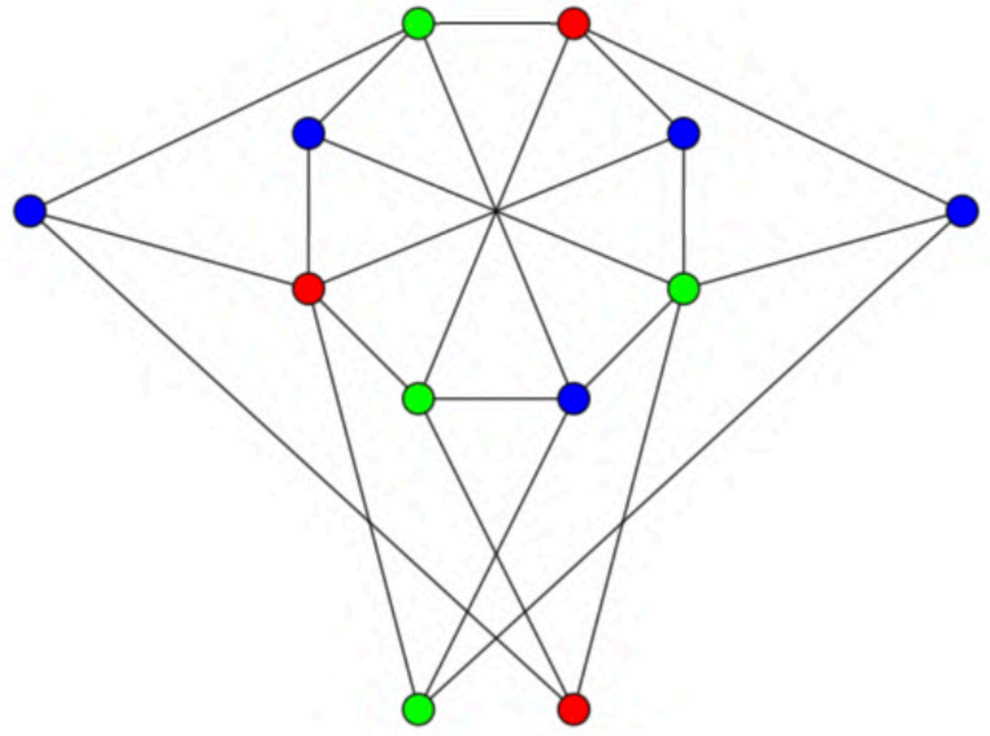

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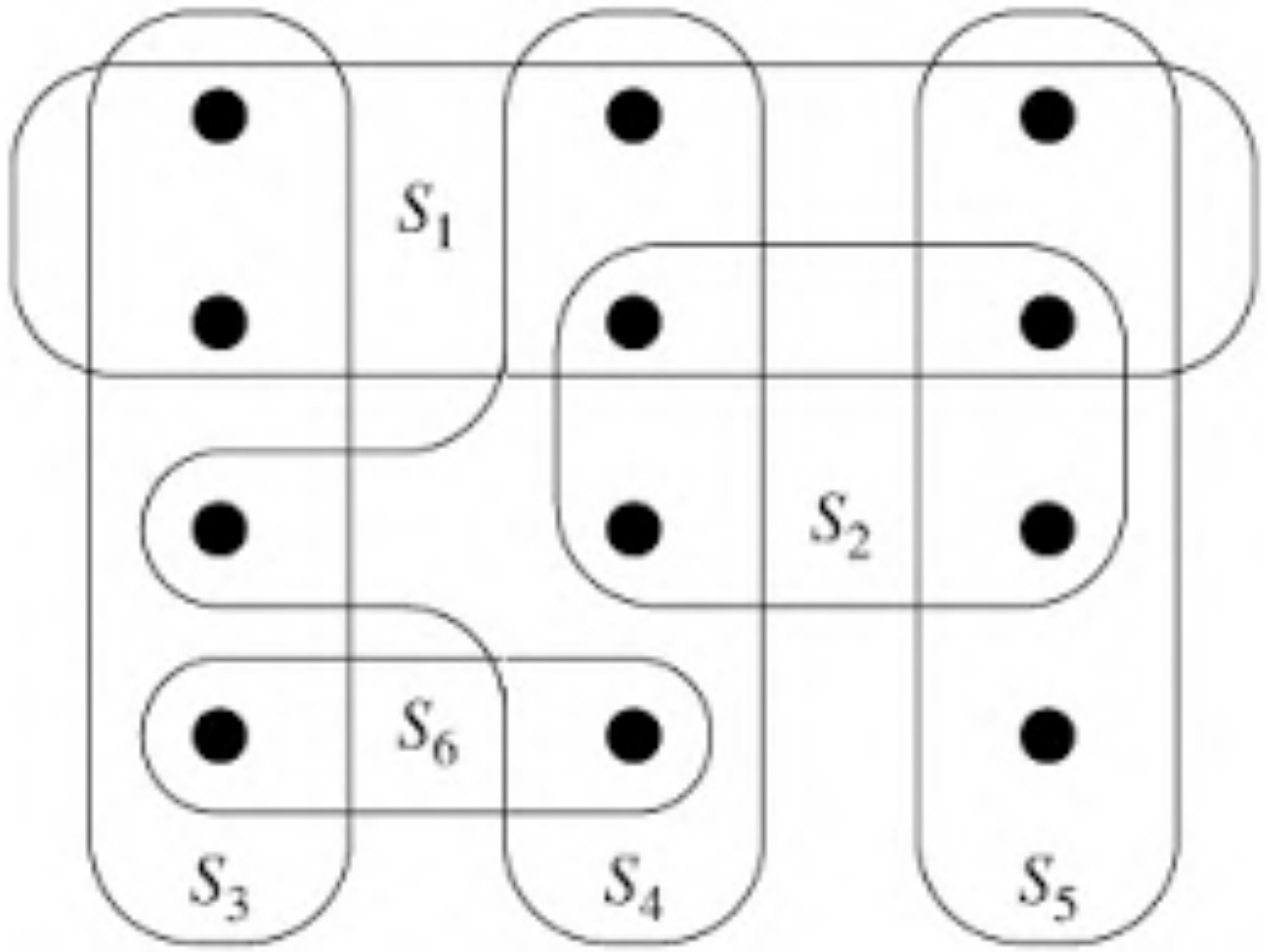

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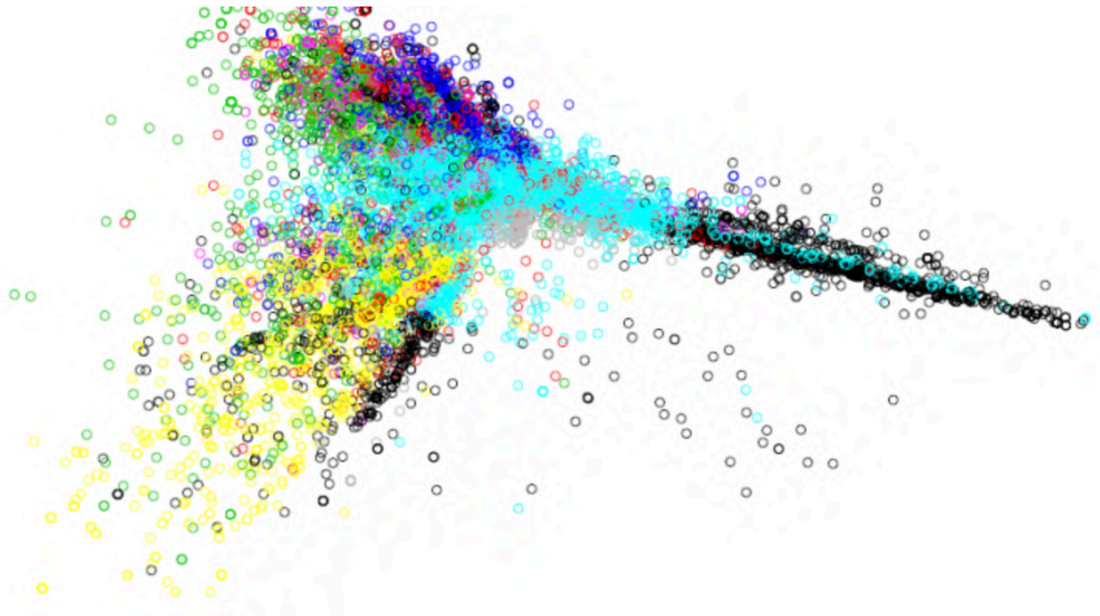

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What about FPT Approximation?

FPT Approximation

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- **Input**: Graph $G = ([k], E)$ and constraints: $\mathscr{C} = \{C_e \subseteq [n] \times [n]\}_{e \in E}$
- **Completeness**: $\exists \sigma : [k] \to [n]$

$$\Pr_{(u,v)\sim E}[(\sigma(u), \sigma(v)) \in C_{(u,v)}] \geq c$$

- **Soundness**: $\forall \sigma : [k] \to [n]$

$$\Pr_{(u,v)\sim E}[(\sigma(u), \sigma(v)) \in C_{(u,v)}] < s$$

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For c=1, s=1, we get exact versions
But we are interested when c=1
and s is bounded away from 1

$$\frac{\quad}{n} \geq c$$

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- **Input**: Graph $G = ([k], E)$ and co...
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Parameterized Inapproximability Hypothesis (PIH) [Lokshtanov-Ramanujan-Saurabh-Zehavi'17]

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2-CSP → (Tight) → k-max-coverage → (Tight) → k-median

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Parameterized Inapproximability Hypothesis (PIH) [Lokshtanov-Ramanujan-Saurabh-Zehavi'17]

ETH ⟹ PIH [Guruswami-Lin-Ren-Sun-Wu'24]

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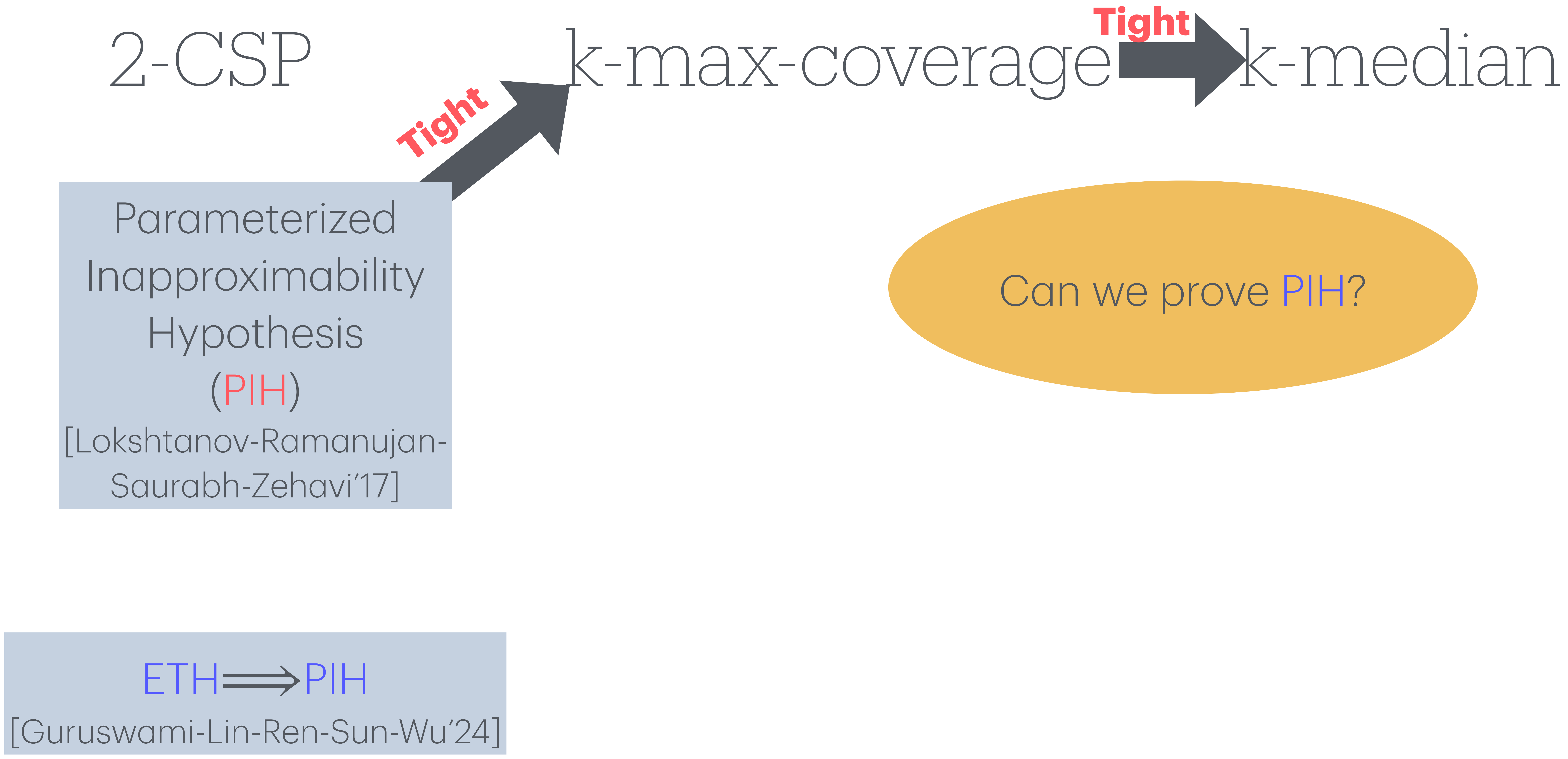

FPT Approximation

2-CSP → (Tight) → k-max-coverage → (Tight) → k-median

Parameterized Inapproximability Hypothesis (PIH) [Lokshtanov-Ramanujan-Saurabh-Zehavi'17]

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Can we prove PIH?

Can we prove parameterized inapproximability of k-max-coverage circumventing PIH?

FPT Approximation

2-CSP —Tight→ k-max-coverage —Tight→ k-median

[Guruswami-Lin-Ren-Sun-Wu'24]

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Once Upon a time...

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Once Upon a time…

SIMONS FOUNDATION

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SIMONS FOUNDATION

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Once Up ne...

Our Results

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While exact k-max-coverage is W[2]-complete,

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If approximating k-max-coverage
to $1 - \varepsilon$ factor is W[1]-complete,
then PIH is true.

Our Results

While exact k-max-coverage is W[2]-complete, approximating it to any $1 - \frac{1}{F(k)}$ factor is W[1]-complete.

If approximating k-max-coverage to $1 - \varepsilon$ factor is W[1]-complete, then PIH is true.

If approximating k-median to $1 + \varepsilon$ factor is W[1]-complete, then PIH is true.

Our Results

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FPT Reductions

If approximating k-max-coverage to $1 - \varepsilon$ factor is W[1]-complete, then PIH is true.

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If approximating k-max-coverage to $1 - \varepsilon$ factor is W[1]-complete, then PIH is true.

If approximating k-median to $1 + \varepsilon$ factor is W[1]-complete, then PIH is true.

Approximating k-max-coverage to $1 - \varepsilon$ factor is W[1]-hard,
$\implies$ Approximating k-max-coverage to $1 - \frac{1}{e} + o(1)$ factor is W[1]-hard

Gap Amplification

If approximating k-max-coverage to $1 - \varepsilon$ factor is W[1]-complete, then PIH is true.

Step 1

Step 2

Step 3

$(\tau, (1-\delta)\cdot\tau)$ k-max-coverage on universe $[n]$

↓

$(\tau', (1-\varepsilon)\cdot\tau')$ k-max-coverage on universe of size $O_\delta(k \log n)$

τ' and ε are functions of δ

If approximating k-max-coverage to $1-\varepsilon$ factor is W[1]-complete, then PIH is true.

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$(c, (1-\varepsilon)\cdot c)$ Valued 2-CSP

$c = O_k(\tau')$

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$\left(1, 1-\frac{\varepsilon}{2}\right)$ 2-CSP

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Randomly hash $[n]$ to universe of size $O_\delta(k \log n)$

If approximating k-max-coverage to $1 - \varepsilon$ factor is W[1]-complete, then PIH is true.

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$(c, (1-\varepsilon)\cdot c)$ Valued 2-CSP

$c = O_k(\tau')$

Partition Universe to $U_1 \dot\cup \cdots \dot\cup U_M$ where $|U_i| = \dfrac{\log n}{\log k}$

Variable x_i is assigned $f : U_i \to [k]$

Variable y_j is assigned input set S_j

Constraints between x_i and y_j measure fraction of U_i mapped to j and covered by S_j

If approximating k-max-coverage to $1 - \varepsilon$ factor is W[1]-complete, then PIH is true.

Step 3

$(c, (1 - \varepsilon) \cdot c)$ Valued 2-CSP

$\Downarrow$

$\left(1, 1 - \frac{\varepsilon}{2}\right)$ 2-CSP

Values of a constraint can take one of $1 + \dfrac{\log n}{\log k}$ entries

There are at most $\binom{k}{2}$ constraints, so we can enumerate all possibilities

Our Results

While exact k-max-coverage is W[2]-complete, approximating it to any $1 - \frac{1}{F(k)}$ factor is W[1]-complete.

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[**K**-Laekhanukit-Manurangsi'19]

If approximating k-max-coverage to $1 - \varepsilon$ factor is W[1]-complete, then PIH is true.

Adjusting parameters in prior works

If approximating k-median to $1 + \varepsilon$ factor is W[1]-complete, then PIH is true.

[Cohen-Addad-Gupta-Kumar-Lee-Li'19]

Thank you for engaging!