New Arenas in Hardness Amplification

Karthik C. S.
(Weizmann Institute of Science)

Joint work with

Elazar Goldenberg
(The Academic College of Tel Aviv-Yaffo)

Necessity is the Mother of Invention

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- Average Case Complexity

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  - Hardness of Problems in Practice

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Modest Goal: Hardness Amplification

Mild average case $\Rightarrow$ Sharp average case

The Utopic Theorem of Hardness Amplification

- Family of functions $\{f_n\}_{n \in \mathbb{N}}$

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- f is "interesting"

The Big Question

Can we do hardness amplification

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Can we do hardness amplification
for problems
we care about and
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The Story so far

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- EXP (Trevisan-Vadhan'07):If $\exists \Pi \in$EXP:
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  then $\exists \Lambda \in$EXP:
  - cannot be efficiently solved on random instances
  - noticeably better than guessing the answer at random.

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The Verona, Pompeii, Flavian, and Fiesole arenas
may not be as well known as the Colosseum,
but are just as impressive.

– *Roman history trivia*

Arenas in Hardness Amplification

NP

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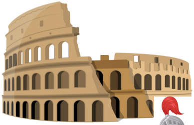

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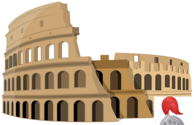

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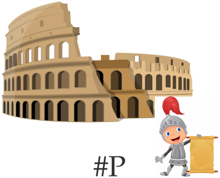

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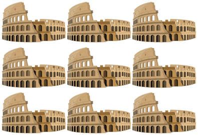

Optimization Problems

Focus of this Talk

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- Subquadratic-hard problems

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Optimization Problems

- ◎ NP-hard problems
- ◎ Subquadratic-hard problems
- ◎ Total Problems

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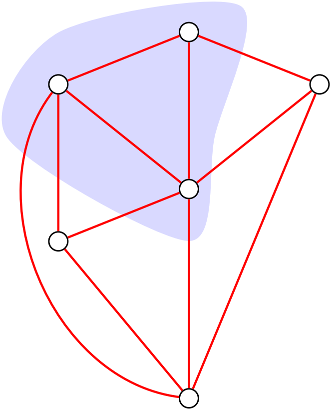

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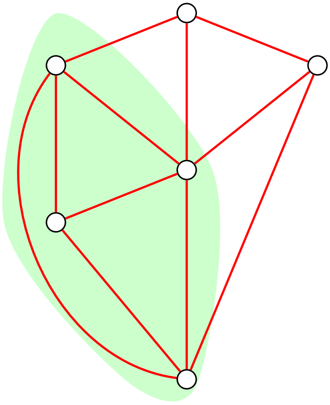

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Then there is $\mathscr{D}'$ a poly(n) time samplable distribution over graphs on poly(n) vertices such that for every randomized algorithm $\mathscr{A}'$ running in time poly(n), we have:

$$\Pr_{G' \sim \mathscr{D}'} \left[\mathscr{A}' \text{ finds max-clique in } G' \text{ w.p. } \geq 2/3 \right] \leq 0.01.$$

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   then $\mathscr{A}$ is correct on $1 - 1/n$ fraction of inputs

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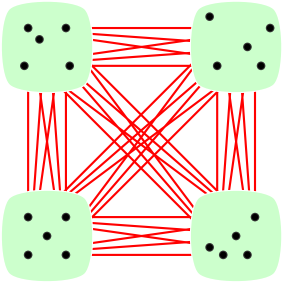

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Sampling time: poly(n)

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3. Output the largest clique in `Solution`

Structure of Optimal Solutions

Claim

If S is a maximum clique of H then for any $i \in [k]$ its restriction to vertices of G_i gives a maximum clique of G_i.

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- Suffices to show: $\mathscr{A}'$ outputs maximum clique in Step 2.5 w.p. ε on $1 - 1/n$ fraction of samples from $\mathscr{D}$.

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$f(x^k) = 1 \iff \mathcal{A}'$ outputs maximum clique w.p. $2/3$

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- Invoke Feige-Kilian lemma to show amplification of hardness

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- $\mathsf{goal}_\Pi \in \{\min, \max\}$.

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- Dec:
  - Input: $i \in [k], x_1, \ldots, x_k \in \mathsf{I}_\Pi(n)$, and optimal $y' \in \mathsf{Sol}_\Pi(x')$
  - Output: optimal $y \in \mathsf{Sol}_\Pi(x_i)$

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- Gen:
    - Input: $x_1, \ldots, x_k \in \mathsf{I}_\Pi(n)$
    - Output: $x' \in \mathsf{I}_\Pi(S(n,k))$
- Dec:
    - Input: $i \in [k]$, $x_1, \ldots, x_k \in \mathsf{I}_\Pi(n)$, and optimal $y' \in \mathsf{Sol}_\Pi(x')$
    - Output: optimal $y \in \mathsf{Sol}_\Pi(x_i)$
- Gen and Dec run in $T(n,k)$ time.

Our General Result

Theorem (Goldenberg-K'19)

Let Π be (S, T)-direct product feasible. Let D be $s(n)$ time samplable distribution over $\mathsf{I}_{\Pi}(n)$ such that for every randomized algorithm $\mathscr{A}$ running in time $t(n)$, we have:

$$\Pr_{x \sim D}\left[\mathscr{A} \text{ finds optimal solution of } x \text{ w.p.} \geq 2/3\right] \leq 1 - \frac{1}{p(n)}.$$

*Conditions apply.

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Then for $k = \text{poly}(p(n))$ there is D' a $\tilde{O}(k \cdot s(n) + T(n, k))$ time samplable distribution over $\mathsf{I}_\Pi(S(n, k))$ such that for every randomized algorithm $\mathscr{A}'$ running in time* $\tilde{O}(t(n))$, we have:

$$\Pr_{x' \sim D'}\left[\mathscr{A}' \text{ finds optimal solution of } x' \text{ w.p. } \geq 2/3\right] \leq 0.01.$$

*Conditions apply.

Problems in P

Theorem (Goldenberg-K'19)

Let D be $\tilde{O}(n)$ time samplable distribution over LCS/Edit Distance such that for every randomized algorithm $\mathscr{A}$ running in time $n^{2-\varepsilon}$, we have:

$$\Pr_{x \sim D}\left[\mathscr{A} \text{ finds optimal alignment of } x \text{ w.p.} \geq 2/3\right] \leq 1 - {}^{1}/{}_{n^{o(1)}}.$$

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Hardness Amplification for

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Connection to Max-SAT

Theorem (Goldenberg-K'19)

Let D be $\mathrm{poly}(n)$ time samplable distribution over Max-SAT such that for every randomized algorithm $\mathscr{A}$ running in time $2^{o(n)}$, we have:

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Can be extended to Vertex Cover, Dominating Set, etc

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Can even be extended to Knapsack,
and other maximization problems!

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Connection to TFNP

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- Find x such that $P(S(x)) \neq x$ or $S(P(x)) = x \neq 0^n$
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- Dec restricts on the corresponding block

Open Problem 1

Average case hard problems in P

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- Can we show some natural problem in P is hard for the uniform distribution?

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Average case hard problems in P

- Can we show some natural problem in P is hard for the uniform distribution?
- Can we construct a fine-grained one way function from worst case assumptions?

Gap Amplification vs. Hardness Amplification

Open Problem 2

Gap Amplification vs. Hardness Amplification

- Can we obtain a trade-off between gap and hardness?

Open Problem 2

Gap Amplification vs. Hardness Amplification

- Can we obtain a trade-off between gap and hardness?
- Can we say something stronger about Max-SAT assuming Gap-ETH?

Open Problem 3

Direct Product Feasibility

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- Can we characterize direct product feasible pairs?

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- Can we show Orthogonal Vectors is self direct product feasible?
- Can we show LCS is self direct product feasible?

Key Takeaways

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  - Amplify hardness from $1/n^{o(1)}$ to $1 - o(1)$
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- Hardness Amplification Technique
  - for Optimization problems
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  - against Randomized algorithms
- Hardness Amplification meets Fine-Grained Complexity
  - Amplify hardness from $1/n^{o(1)}$ to $1 - o(1)$ for LCS, Edit Distance, etc.
  - If ETH is true on mild worst case then Max-SAT is hard on average

THANK
YOU!