Inapproximability of Clustering in ℓ_p-metrics

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Joint work with

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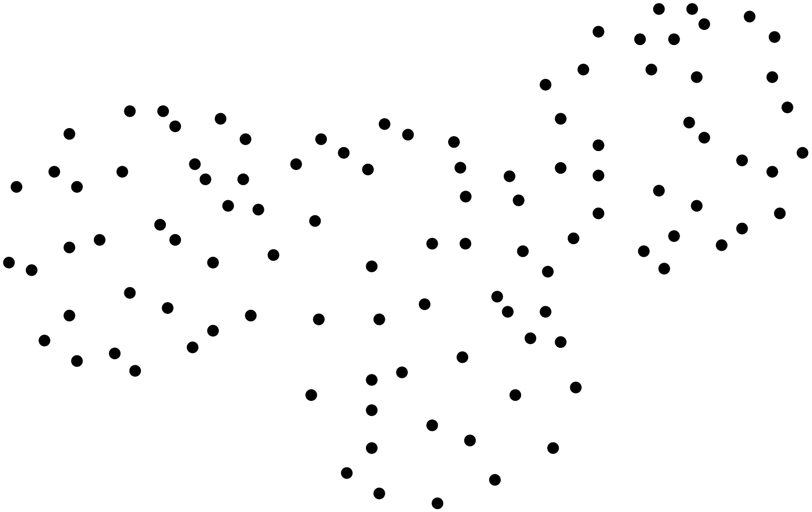

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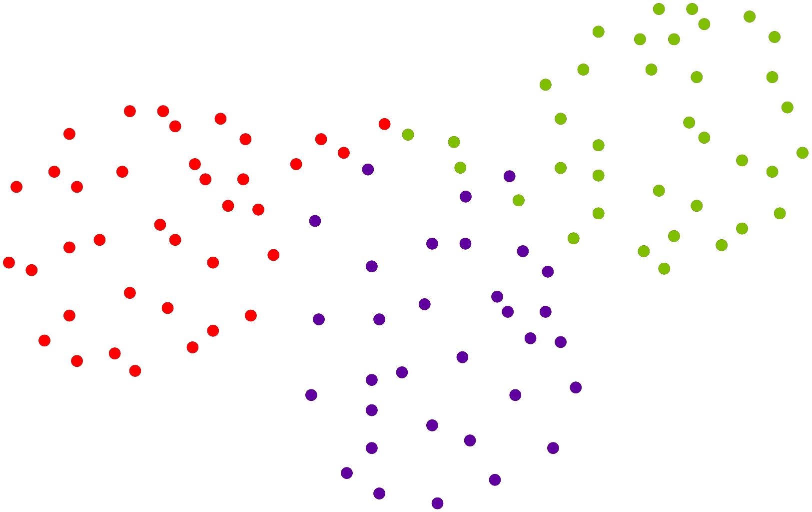

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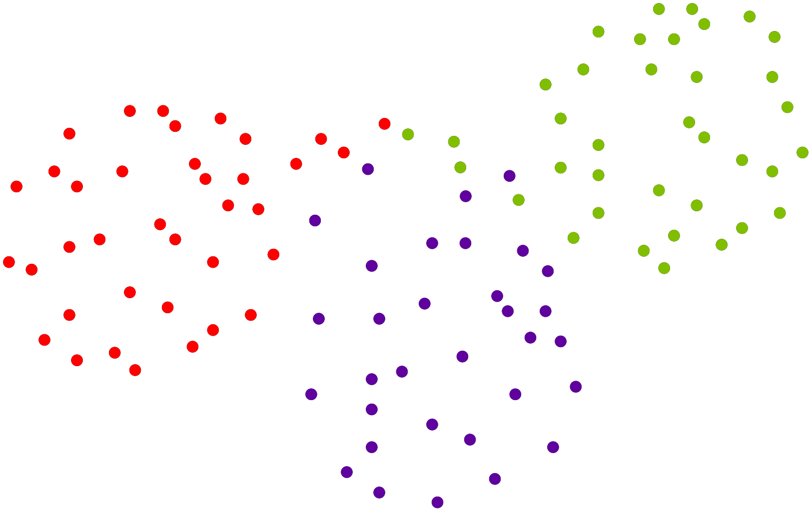

Task of Classifying Input Data

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k-means in Euclidean metric < 1.0013
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A New Embedding Framework to potentially get Strong (tight?) Inapproximability results!

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Edges $\rightarrow$ Data Points

Vertices $\rightarrow$ Candidate Centers

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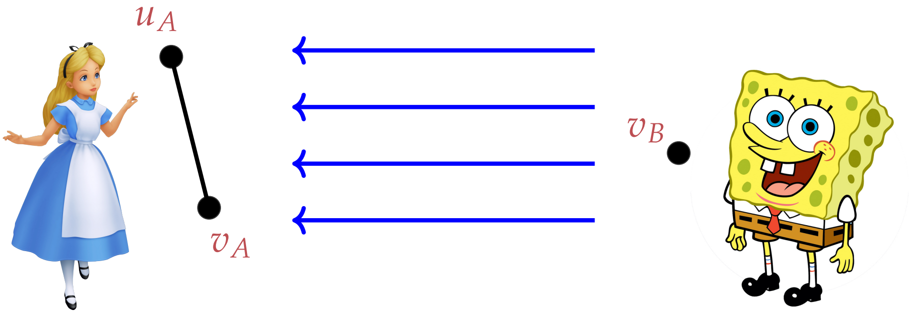

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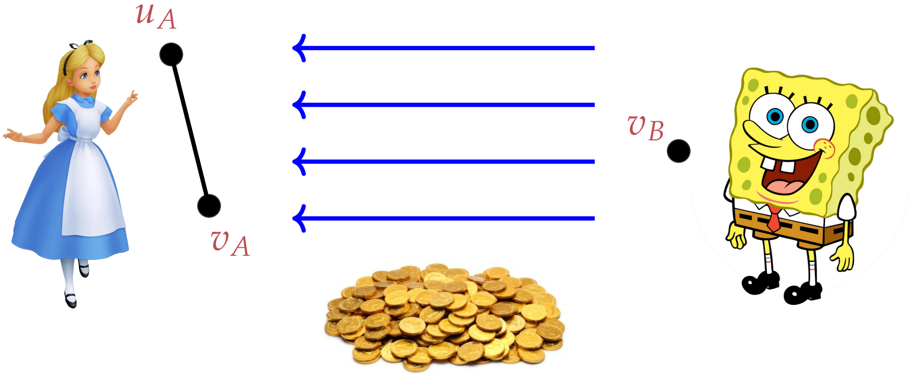

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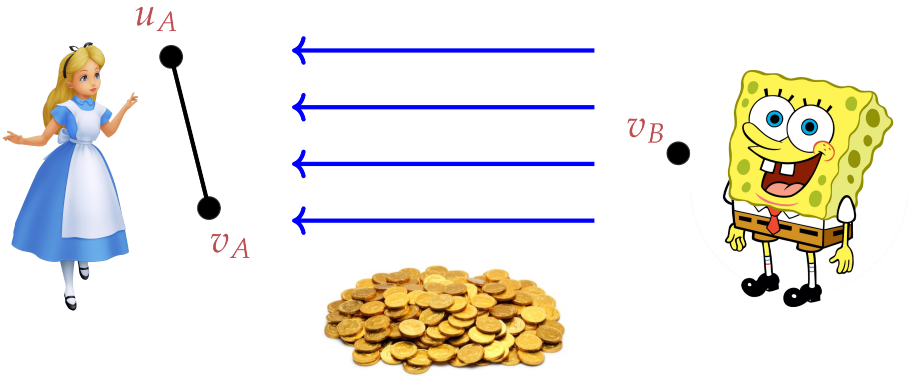

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GOAL

$$\boxed{\text{Determine if } v_B \in \{u_A, v_A\}}$$

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- k-means objective is:

$$\sum_{x \in X} \|(x - \sigma(x)\|_0^2 = (c \cdot \log n)^2 \cdot |X \setminus X_{E'}| + 9 \cdot (c \cdot \log n)^2 \cdot |X_{E'}|$$

Theorem (Cohen-Addad–K'19)

Given input $X, \mathcal{S} \subseteq \{0,1\}^{O(\log n)}$. It is UG-hard to distinguish:

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NO: For all (C, σ) we have $\sum_{x \in X} \|(x - \sigma(x)\|_0^2 \geq 1.56 \cdot n'$,

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k-means in Hamming metric: Continuous Case

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NO: For all (C, σ) we have $\sum_{x \in X} \|(x - \sigma(x)\|_0^2 \ge 1.21 \cdot n'$,

where $n' = O(n(\log n)^2)$.

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  - Mostly 1 $\Rightarrow$ decode vertex

Our Result

Discrete Version

	k-means	k-median
ℓ_1-metric	1.56	1.14
ℓ_2-metric	1.17	1.06
ℓ_∞-metric	3.94	1.74

Continuous Version

k-means in ℓ_2-metric ≈ 1.07

k-median in ℓ_1-metric ≈ 1.07

Other Metrics: More Embedding

Gap Number of ℓ_p-metric

Largest $\alpha > 1$ for which we can realize $V \cup E$ of K_n such that

$$\|u - e\|_p = 1 \text{ if } u \in e \quad \text{and} \quad \|u - e\|_p \geq \alpha \text{ if } u \notin e$$

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Replace each block by the embedding realizing gap number

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  - Argue that # of edges in cluster $\gg$ max degree of cluster

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For every $\delta > 0$ there is some $h \in \mathbb{N}$ such that deciding an instance of $(1 - 1/e + \varepsilon)$-hypergraph vertex coverage problem on h-uniform hypergraphs is NP-hard.

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Gap hypergraph number in ℓ_∞-metric is 3

Key Takeaways

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- Current Reduction gives gap number $1 + 2/(h-1)$
- Dimension of embedding doesn't matter for ℓ_2-metric
  - Johnson-Lindenstrauss dimension reduction

Open Problem 2

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Can we embed vertices and edges of K_n
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- It holds for $n = 3$
- Can we prove an upper bound of 2?

Open Problem 3

Can we go beyond Triangle Inequality Barrier?

Open Problem 3

Can we go beyond Triangle Inequality Barrier?

- Can we show $>1 + 8/e$ inapproximability of k-means in any metric?
- Can we show $>1 + 2/e$ inapproximability of k-median in any metric?

THANK
YOU!