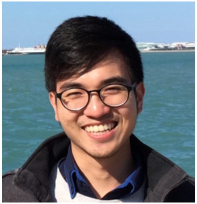
On Complexity of Closest Pair Problem

Karthik C. S.
(Weizmann Institute of Science)

Joint work with



Pasin Manurangsi
(UC Berkeley)

Part I

The Bird's Perspective

Closest Pair

- Closest Pair problem (CP) in ℓ_p-metric

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  What happens when $d = \omega(1)$?

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For every $\varepsilon > 0$, there exists $k(\varepsilon) \in \mathbb{N}$, such that no algorithm running in $2^{m(1-\varepsilon)}$ time can solve k-SAT on m variables.

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- BCP in ℓ_p-metric when $d = 2^{O(\log^* n)}$ [Williams'18, Chen'18].

Equivalence of Bichromatic Closest Pair and Closest Pair

BCP is at least as hard as CP in every ℓ_p-metric for all d.

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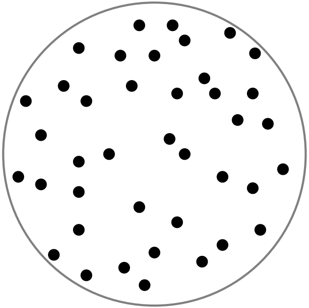

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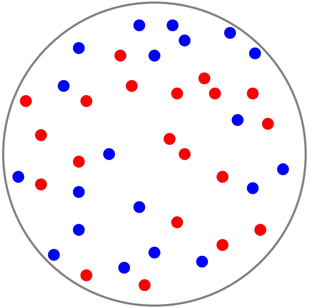

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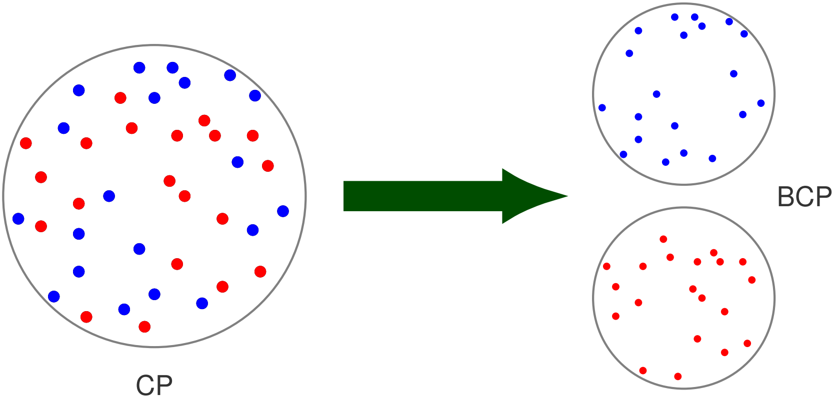

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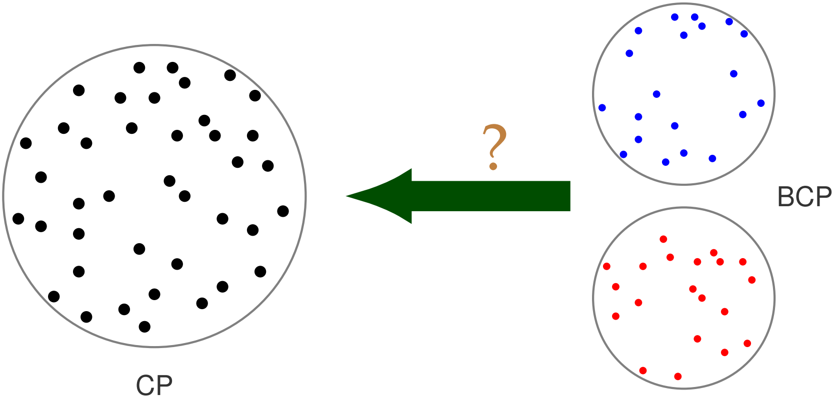

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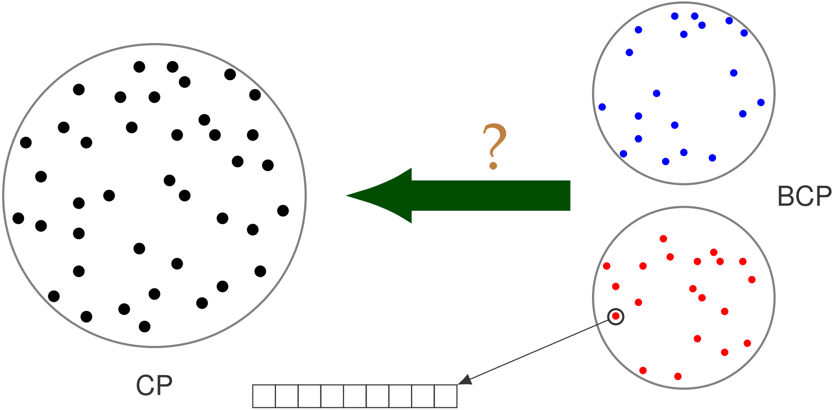

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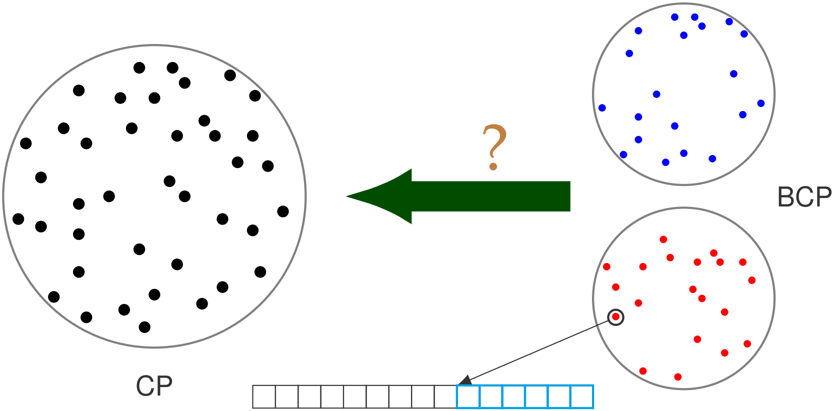

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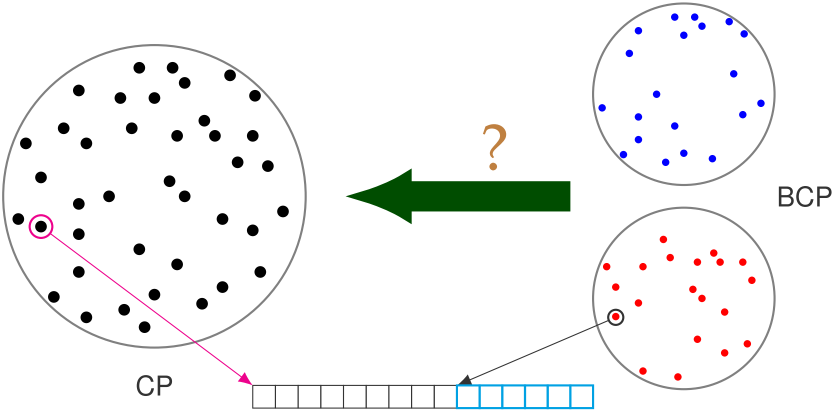

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Theorem (David-K-Laekhanukit'18)

CP is as hard as BCP in ℓ_p-metric when $d = \Omega(\mathrm{cd}_p(K_{n,n}))$.

Contact Dimension of a Graph

Contact Dimension of a Graph ($\mathrm{cd}_p(G)$)

Smallest dimension for which we can realize:

$$\|u - v\|_p = 1 \text{ if } (u, v) \in G \quad \text{and} \quad \|u - v\|_p > 1 \text{ otherwise}$$

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- $\text{cd}_p(K_{n,n}) = \Theta(\log n)$ for $p > 2$ [David-K-Laekhanukit'18]
- $\text{cd}_2(K_{n,n}) = \Theta(n)$ [Maehara'91]

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- We build $(n, d + d^*, A' \cup B')$ instance of CP in ℓ_p-metric

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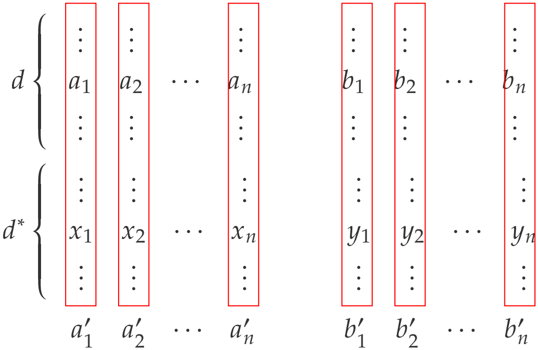

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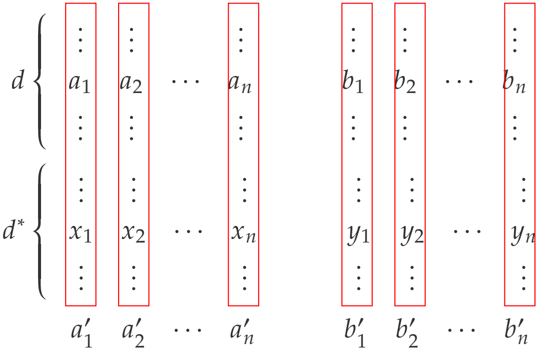

Points from same set:

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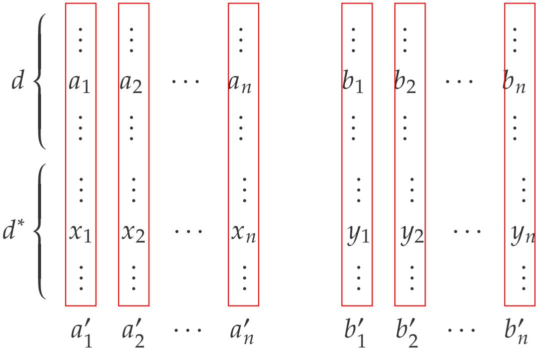

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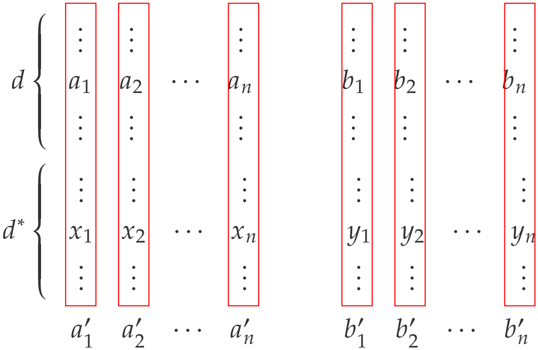

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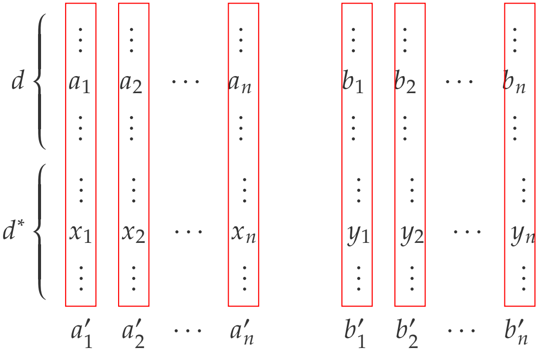

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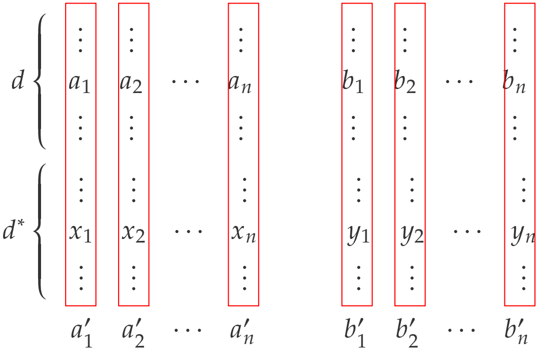

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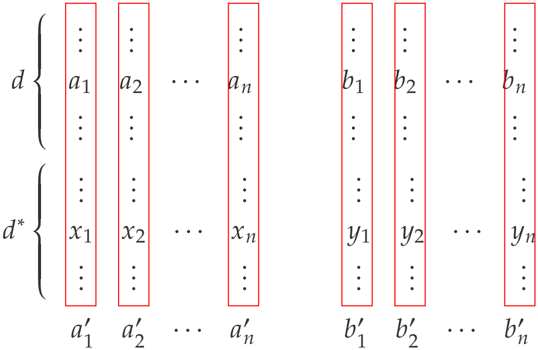

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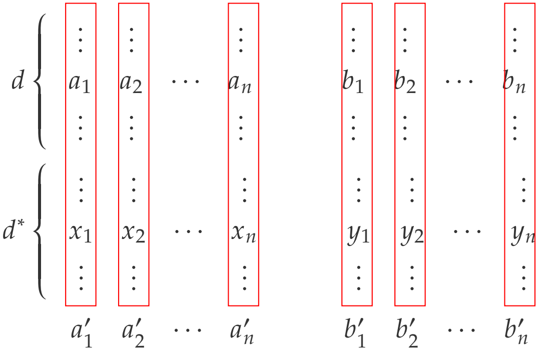

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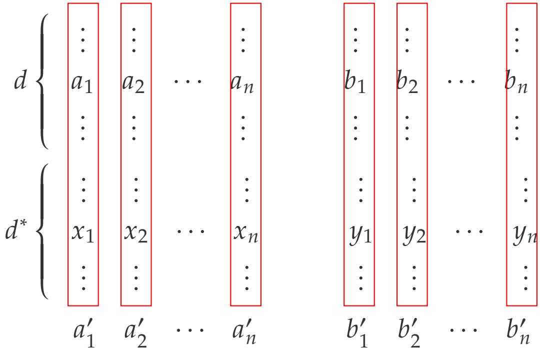

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Let $p > 2$. Assuming SETH, for every $\varepsilon > 0$, no $n^{2-\varepsilon}$ time algorithm can solve:

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Raised as open question recently:

- Abboud-Rubinstein-Williams'17
- Williams'18
- David-K-Laekhanukit'18

Theorem (K-Manurangsi'18)

Let $p \geqslant 1$. Assuming SETH, for every $\varepsilon > 0$,

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- no $n^{1.5-\varepsilon}$ time algorithm can solve $(1+\delta)$-approximate CP in ℓ_p-metric when $d = \omega(\log n)$.

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- $E(G_1) \cup E(G_2) \cup \cdots \cup E(G_k) = E(K_{n,n})$
- $\forall r \in [k]$, $\mathsf{cd}_p(G_r) = \text{polylog } n$, $\forall p \geqslant 1$

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- Let (a_{i^*}, b_{j^*}) be solution to BCP
- There exists $r^* \in [k]$, $(i^*, j^*) \in E(G_{r^*})$
- Then (a'_{i^*}, b'_{j^*}) in r^* instance $(n, d + d^*, A'_{r^*} \cup B'_{r^*})$ of CP is a solution

Dense Bipartite Graph with Low Contact Dimension

We find $G_1, \ldots, G_k$ on $V(K_{n,n})$ where $k = n^{o(1)}$ such that:

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$$f \in \mathbb{F}_q \longrightarrow (0, \ldots, 0, 1, 0, \ldots, 0) \in \{0,1\}^q$$

f^{th} position

New Agenda

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Contact Dimension of a Random Graph is $\Omega(n)$

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Polynomials are our friends.

– TCS Folklore

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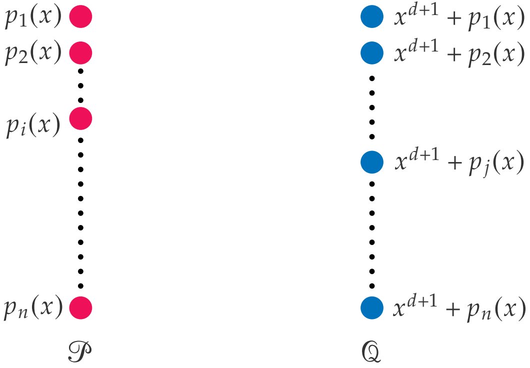

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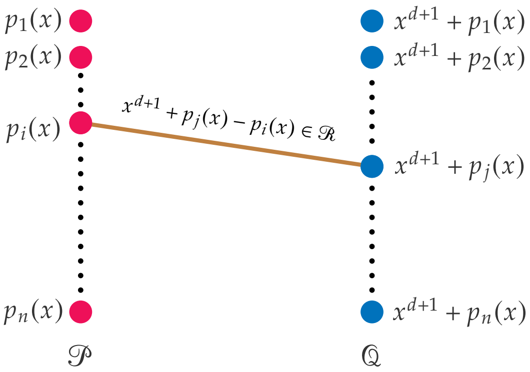

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Proof Summary

- A bipartite graph G^* on $V(K_{n,n})$:
  - **Dense**: $|E(G^*)| = n^{2-o(1)}$
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(A, B, n, d)
BCP instance

$\Rightarrow$

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CP instances

Proof Summary

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} CP instances

Theorem (K-Manurangsi'18)

Let $p \geqslant 1$. Assuming SETH, for every $\varepsilon > 0$,

- no $n^{2-\varepsilon}$ time algorithm can solve CP in ℓ_p-metric when $d = (\log n)^{\Omega_\varepsilon(1)}$.
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Key Takeaways

For every $p \geqslant 1$,

- Closest Pair problem in ℓ_p-metric cannot be solved[1] in subquadratic (in n) time when $d = (\log n)^{\Omega(1)}$.

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There is a dense bipartite graph with low contact dimension

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Part II

The Frog's Perspective

Fine-Grained Complexity of Closest Pair

Theorem

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If $|B(s^*, r^*) \cap C^*| = n^{\delta}$: Locally Dense Codes

(Dumer-Miccancio-Sudan'03)

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- (C^*, s^*) can be found in poly(n) time

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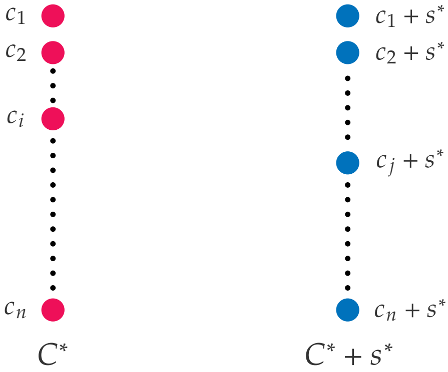

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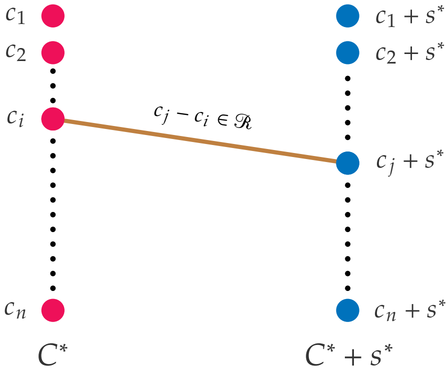

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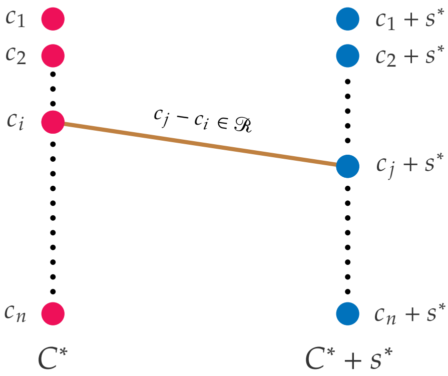

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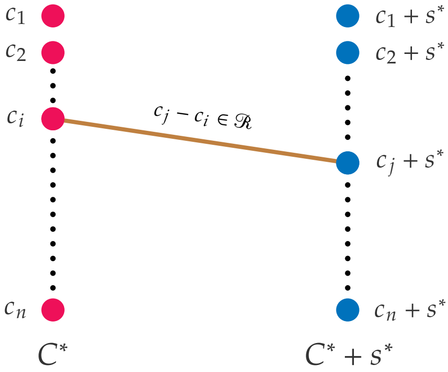

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Contact Dimension of G*: $\mathrm{cd}_0(G^*) = \ell$ over alphabet $\mathbb{F}_q$

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- Reed Solomon codes cannot have large T and give above gap.

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- $T \approx \sqrt{|\widetilde{C}^*|}$ (Ashikhmin-Barg-Vlăduţ'01, Vlăduţ'18)

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Can $(1+\delta)$-CP be solved in $n^{2-\varepsilon}$ time for some $\varepsilon > 0$ and every $\delta > 0$ in $\omega(\log n)$ dimensions?

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- Algebraic Geometric Codes with Better Parameters
- Can construct gap-CP instance in high dimensions
  - Johnson-Lindenstrauss dimension reduction

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Triangle Inequality Barrier for gap-BCP (Rubinstein'18):

Can we show assuming SETH, for some $\varepsilon > 0$, 3-BCP cannot be solved in $n^{1+\varepsilon}$ time in $\omega(\log n)$ dimensions in any metric?

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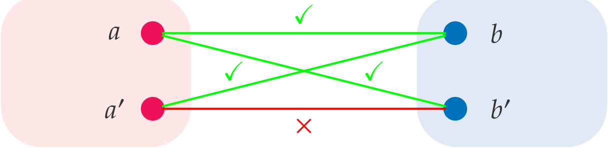

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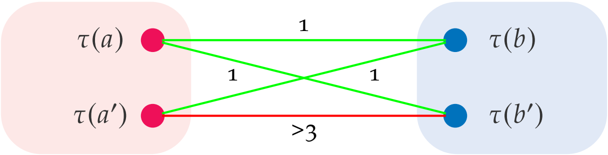

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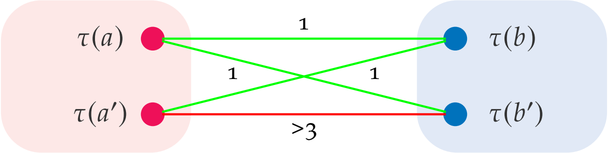

★ Assuming SETH, no subquadratic time algorithm for $(3 - o(1))$-BCP in ℓ_∞-metric (David-K-Laekhanukit'18)

Triangle Inequality Barrier for ~~gap-BOP~~ gap-CP (Rubinstein'18):

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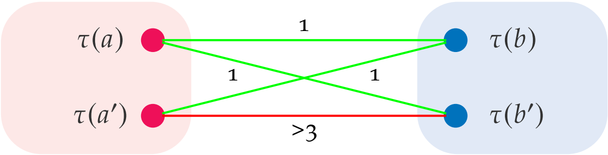

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Open Problem 3

Bichromatic Maximum Inner Product problem (BMIP)

Input: $A, B \subset \mathbb{R}^d$, $|A| = |B| = n$, Output: $a^* \in A, b^* \in B, \max_{\substack{a \in A \\ b \in B}} \langle a, b \rangle$

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Theorem (Abboud-Rubinstein-Williams'17)

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THANK
YOU!