Hardness of Approximation meets Parameterized Complexity

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New York University

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Global Outline

Day 1: The Setting

Day 2: Gap Creation

Day 3: Applications

Day 2 Outline

Part 1: Hardness of Approximating MaxCover

- Recap
- MaxCover with Projection Property
- Gap Creation

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Part 2: Hardness of Approximating One-Sided Biclique

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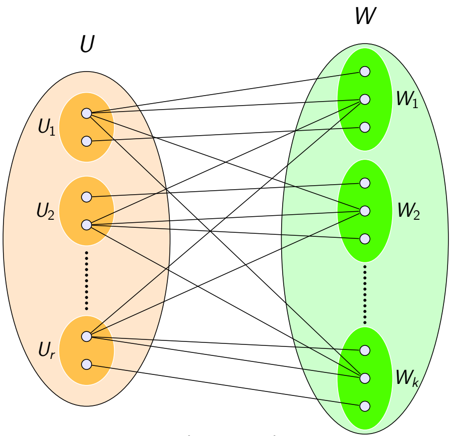
Part 1

Gap Creation in MaxCover

MaxCover: Recap



$\Gamma(U, W, E)$

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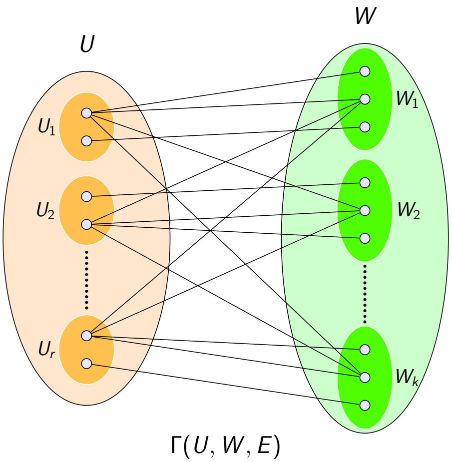

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Each U_i is a Left Super Node

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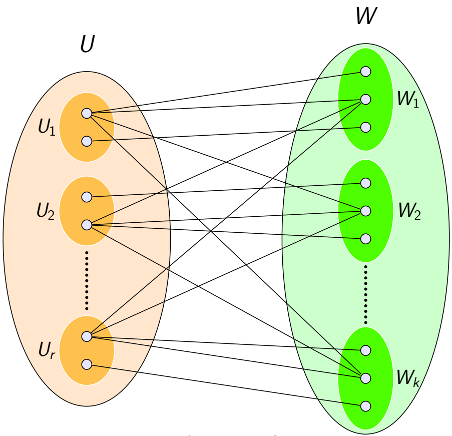

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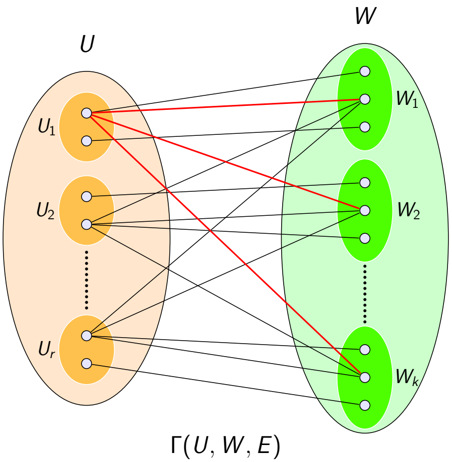

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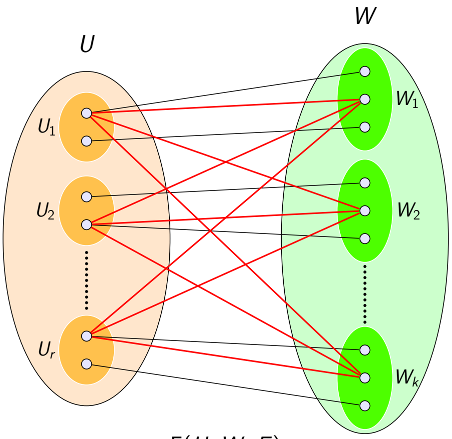

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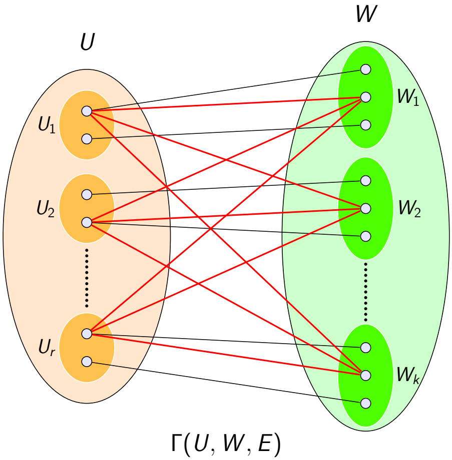

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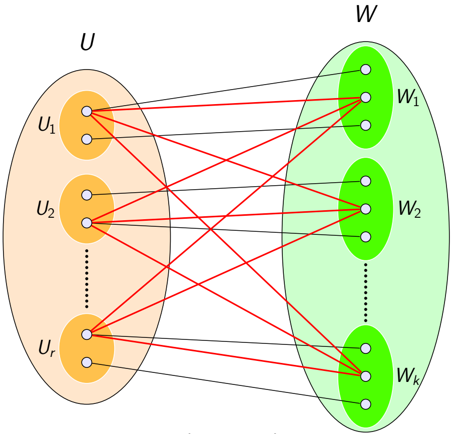
MaxCover: Recap



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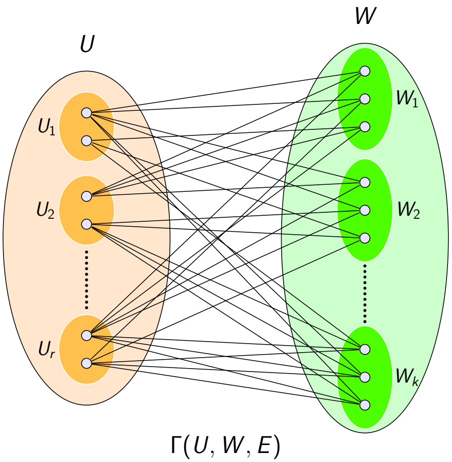
Determine if $\text{MaxCover}(\Gamma) = 1$
or $\text{MaxCover}(\Gamma) \leq s$

MaxCover: Projection Property



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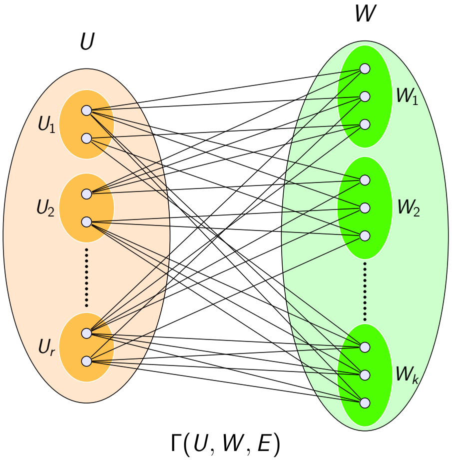

Γ has projection property:

For every U_i and W_j,

Induced subgraph of (U_i, W_j) is:

- complete bipartite graph (i.e., irrelevant), or,
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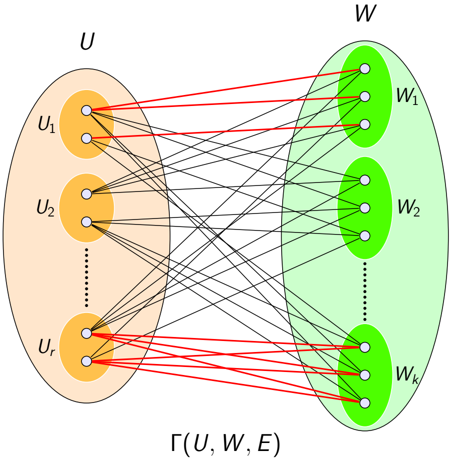

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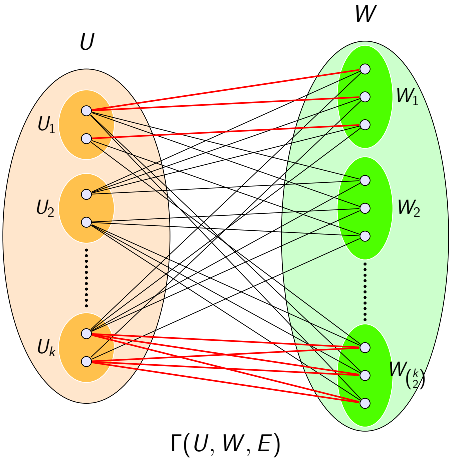
MaxCover with Projection Property is W[1]-Hard



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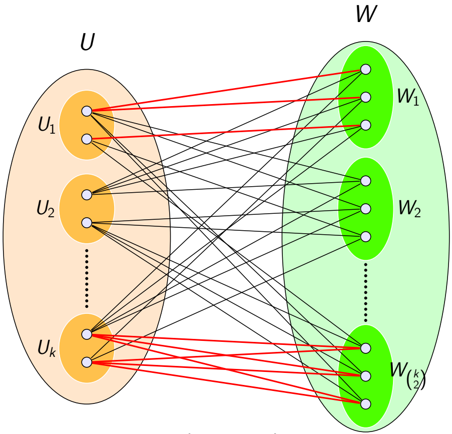
MaxCover with Projection Property is W[1]-Hard



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Input: $G([n], E_0)$

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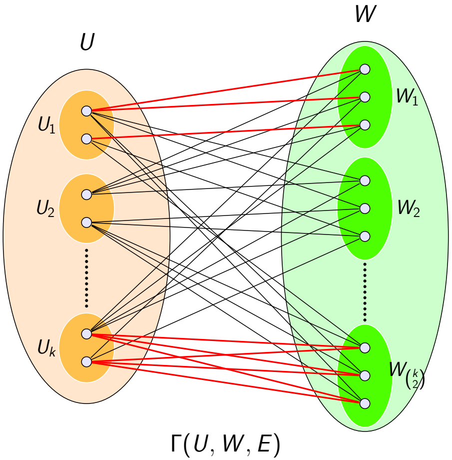

Input: $G([n], E_0)$

$U_i = [n]$ and $W_{j,j'} = E_0$

$W_{j,j'}$ has projection to U_j and $U_{j'}$

MaxCover: Gap Creation

Inapproximability of MaxCover [K-LivniNavon'21]

There is a FPT reduction from MaxCover instance $\Gamma_0 = \left(U_0 = \bigcup_{j=1}^{r} U_j^0, W = \bigcup_{j=1}^{k} W_i, E_0\right)$ with projection property to a MaxCover instance $\Gamma = \left(U = \bigcup_{j=1}^{O(\log |U_0|)} U_j, W = \bigcup_{j=1}^{k} W_i, E\right)$ such that

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- If $\text{MaxCover}(\Gamma_0) = 1$ then $\text{MaxCover}(\Gamma) = 1$
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- $|\Gamma| = \tilde{O}(2^r \cdot |W| \cdot \log |U_0|)$
- The reduction runs in time $2^{O(r)} \cdot \text{poly}(|\Gamma_0|)$.

Coding Theory: Recap

- $C \subseteq [q]^L$
- Distance of C:

$$\Delta(C) := \min_{x,y \in C} \|x - y\|_0$$

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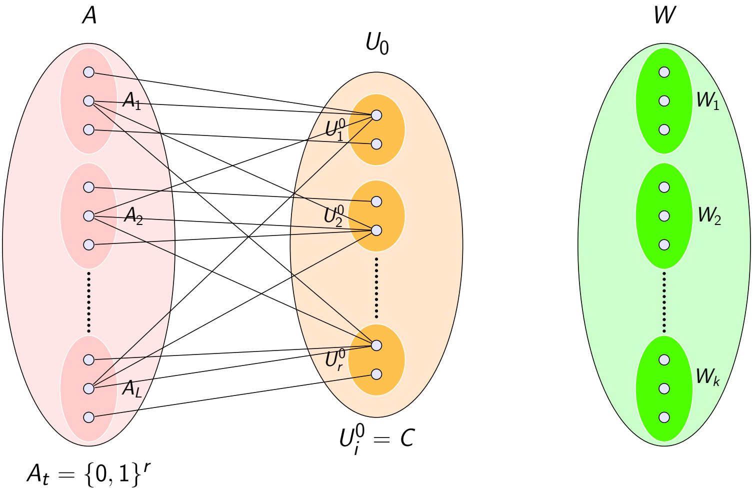
- Reed Solomon Codes:
  - Evaluations of degree d univariate polynomials over $\mathbb{F}_q$
  - $|\mathrm{RS}| = q^{d+1}$
  - $\Delta(\mathrm{RS}) = q - d$
  - q^{d+1} codewords in $[q]^q$ with distance $q - d$

Threshold Graph Construction



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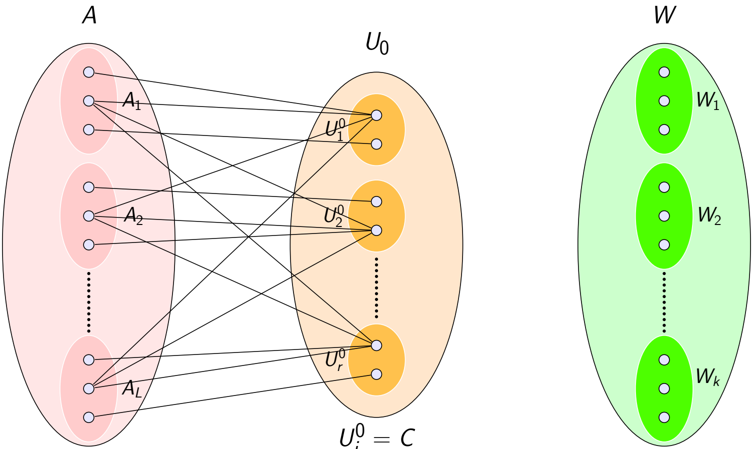

$A_t = \{0,1\}^r$

$U_i^0 = C$

$(u, (q_1, \ldots, q_r)) \in U_i^0 \times A_t$ is an edge $\Leftrightarrow u_t = q_i$

Threshold Graph Properties

Completeness

For every $(u^1, \ldots, u^r) \in U_1^0 \times \cdots \times U_r^0$ and every A_t
there exists a unique common neighbor of $(u^1, \ldots, u^r)$ in A_t

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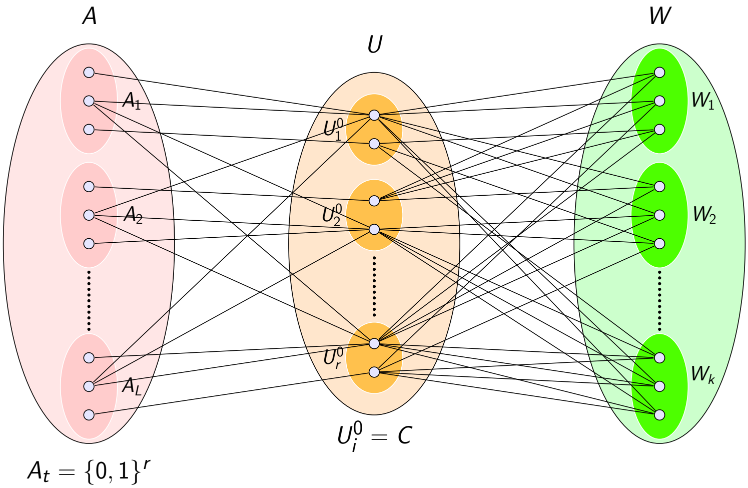
Soundness

For every $u, u' \in U_i^0$, there are at most $L - \Delta(C)$ many supernodes in A
which have a common neighbor of u and u'

Threshold Graph Composition



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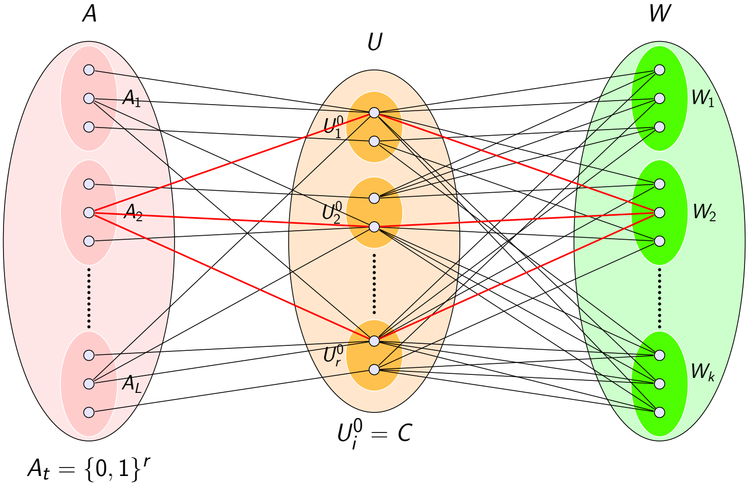

$(w, (q_1, \ldots, q_r)) \in W_j \times A_t$ is an edge $\Leftrightarrow \exists\ (u^1, \ldots, u^r) \in U_1^0 \times \cdots U_r^0$ such that $\forall i \in [k]$, (w, u^i) and $(u^i, (q_1, \ldots, q_r))$ are both edges

Completeness of Reduction

- Let $(w_1, \ldots, w_k) \in W_1 \times \cdots \times W_k$ be optimal labeling of Γ_0
- Let $(u^1, \ldots, u^r) \in U_1^0 \times \cdots \times U_r^0$ be common neighbors of $(w_1, \ldots, w_k)$ in Γ_0

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Completeness of Threshold Graph

For every $(u^1, \ldots, u^r) \in U_1^0 \times \cdots \times U_r^0$ and every A_t
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Inapproximability of MaxCover using Random Binary Codes

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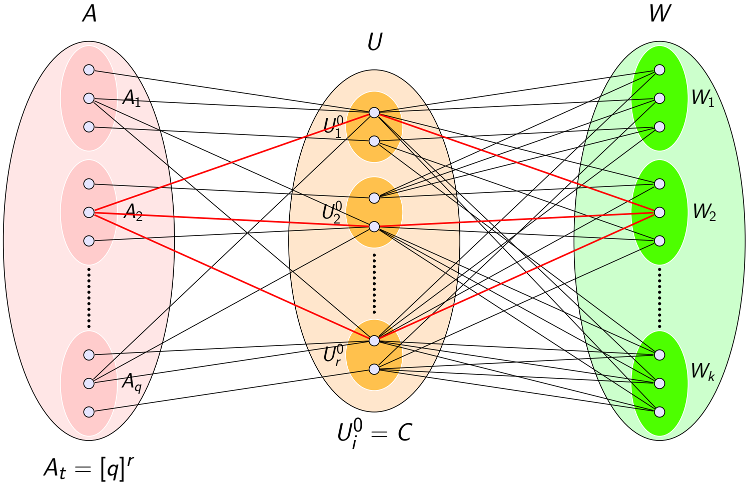
Threshold Graph Composition with Reed Solomon Codes



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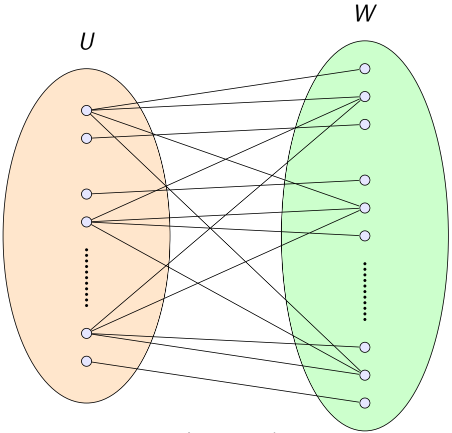
- If $\text{MaxCover}(\Gamma_0) = 1$ then $\text{MaxCover}(\Gamma) = 1$
- If $\text{MaxCover}(\Gamma_0) < 1$ then $\text{MaxCover}(\Gamma) \leq \frac{\log_q |U_0|}{q}$
- $|\Gamma| = \tilde{O}(q^r \cdot |W| \cdot \log |U_0|)$
- The reduction runs in time $q^r \cdot \text{poly}(|\Gamma_0|)$.

Part 2

Gap Creation in One-Sided Biclique

One-Sided Biclique: Recap



$\Gamma(U, W, E)$

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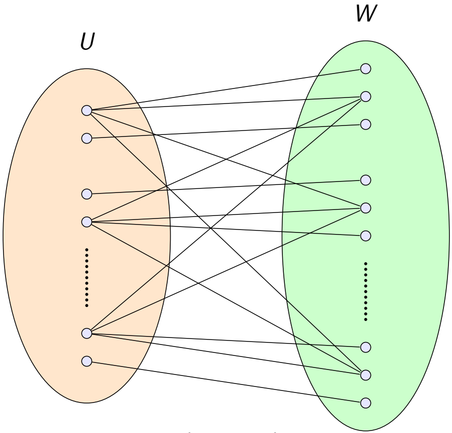
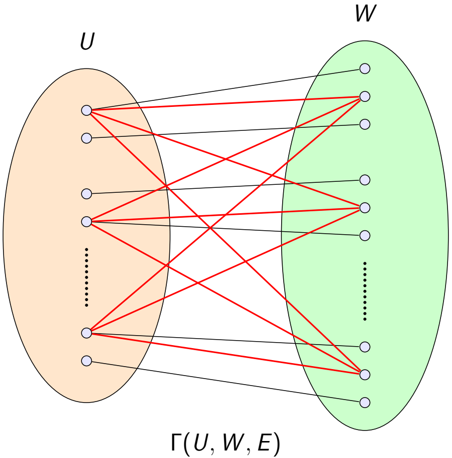


$\Gamma(U, W, E)$

Find k vertices in W
with most common neighbors

One-Sided Biclique: Recap



Find k vertices in W
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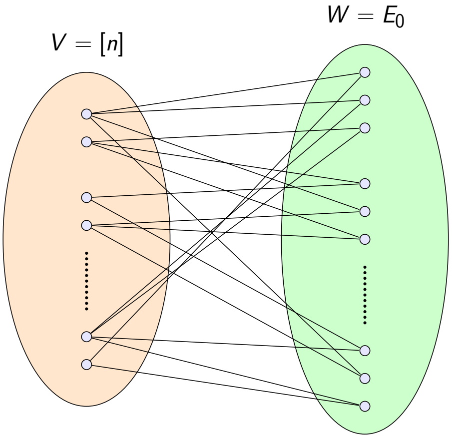
One-Sided Biclique: Gap Creation

Inapproximability of One-Sided Biclique (Lin'18)

There is a FPT reduction from k-Clique instance $G([n], E_0)$ to a One-Sided Biclique instance $\Gamma = (U, W, E)$ such that

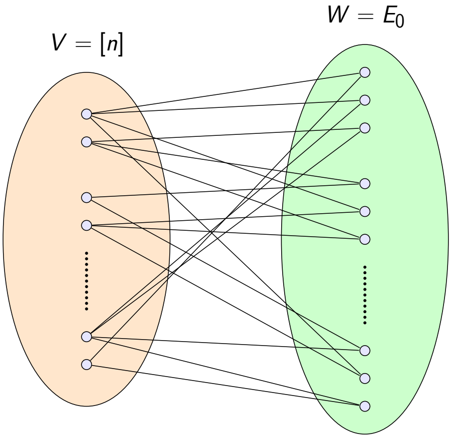
- If G has a k-clique then there are $\binom{k}{2}$ vertices in W which have $n^{1/k}$ common neighbors in U
- If G has no k-clique then for every $\binom{k}{2}$ vertices in W they have at most $(k+1)!$ common neighbors in U
- $|\Gamma| = n^3$
- The reduction runs in time poly(n)

Starting from k-Clique



Input: $G([n], E_0)$

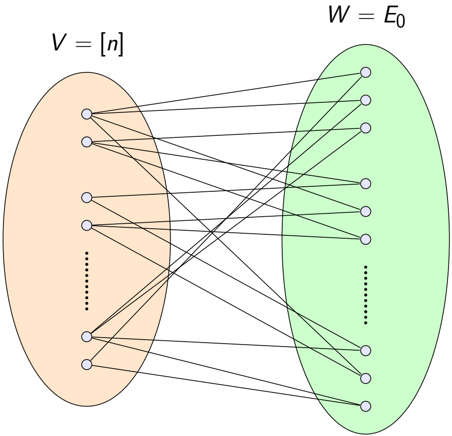
Starting from k-Clique



Input: $G([n], E_0)$

If G has a k-clique then there are $\binom{k}{2}$ vertices in W which in total have k neighbors

Starting from k-Clique

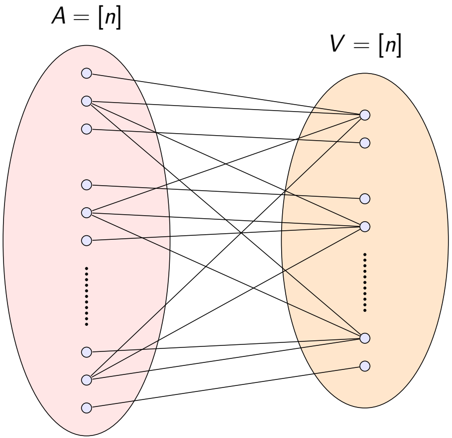


Input: $G([n], E_0)$

If G has a k-clique then
there are $\binom{k}{2}$ vertices in W
which in total have k neighbors

If G has no k-clique then
any $\binom{k}{2}$ vertices in W
has totally at least $k+1$ neighbors

Threshold Graph



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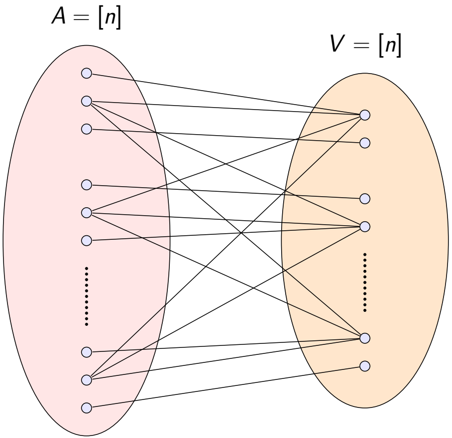

Every k vertices in V has at least $n^{1/k}$ common neighbors in A

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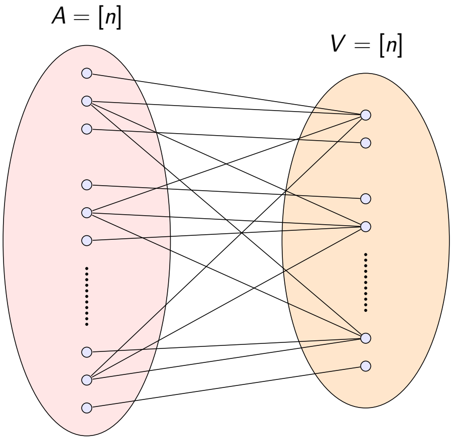

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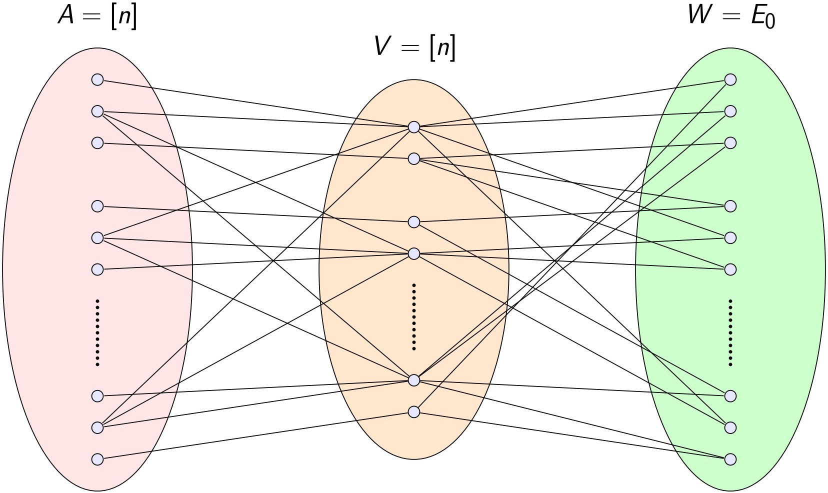
Every $k+1$ vertices in V has at most $(k+1)!$ common neighbors in A

Threshold Graph Composition



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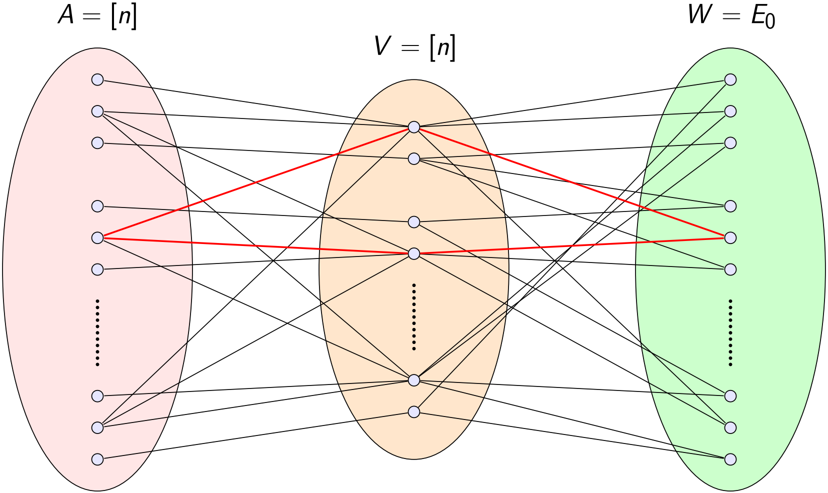

$(w, a) \in W \times A$ is an edge $\Leftrightarrow \exists\ v, v' \in V$ such that a and w are common neighbors of v and v'

Completeness of Reduction

- Let $v_1, \ldots, v_k \in V$ be vertices of k-clique in G
- Let $A' \subseteq A$ be common neighbors of $v_1, \ldots, v_k$ in Threshold graph

Completeness of Reduction

- Let $v_1, \ldots, v_k \in V$ be vertices of k-clique in G
- Let $A' \subseteq A$ be common neighbors of $v_1, \ldots, v_k$ in Threshold graph
- Every $a \in A'$ is also a common neighbor of $e_{v_i,v_j} \in W$ in Γ

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Completeness of Threshold Graph

Every k vertices in V has at least
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Soundness of Reduction

- Fix $(w_1, \ldots, w_{\binom{k}{2}}) \in W$ and let $A' \subseteq A$ be its set of common neighbors in Γ

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- Fix $(w_1, \ldots, w_{\binom{k}{2}}) \in W$ and let $A' \subseteq A$ be its set of common neighbors in Γ
- Let $V' \subseteq V$ be set of total neighbors of $(w_1, \ldots, w_{\binom{k}{2}})$ in V
- $|V'| \geq k+1$

Soundness of Reduction

- Fix $(w_1, \ldots, w_{\binom{k}{2}}) \in W$ and let $A' \subseteq A$ be its set of common neighbors in Γ
- Let $V' \subseteq V$ be set of total neighbors of $(w_1, \ldots, w_{\binom{k}{2}})$ in V
- $|V'| \geq k + 1$
- A' is a subset of the common neighbors of V' in Threshold graph

Soundness of Reduction

- Fix $(w_1, \ldots, w_{\binom{k}{2}}) \in W$ and let $A' \subseteq A$ be its set of common neighbors in Γ
- Let $V' \subseteq V$ be set of total neighbors of $(w_1, \ldots, w_{\binom{k}{2}})$ in V
- $|V'| \geq k+1$
- A' is a subset of the common neighbors of V' in Threshold graph

Soundness of Threshold Graph

Every $k+1$ vertices in V has at most $(k+1)!$ common neighbors in A

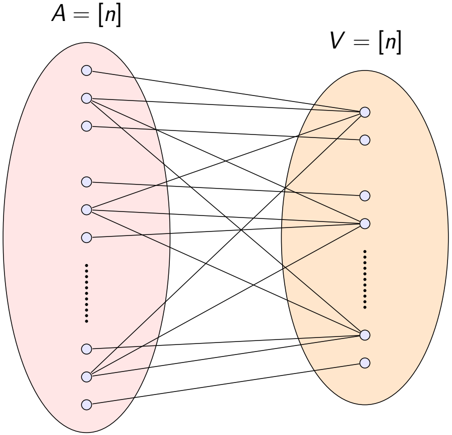
One-Sided Biclique: Gap Creation

Inapproximability of One-Sided Biclique (Lin'18)

There is a FPT reduction from k-Clique instance $G([n], E_0)$ to a One-Sided Biclique instance $\Gamma = (U, W, E)$ such that

- If G has a k-clique then there are $\binom{k}{2}$ vertices in W which have $n^{1/k}$ common neighbors in U
- If G has no k-clique then for every $\binom{k}{2}$ vertices in W they have at most $(k+1)!$ common neighbors in U
- $|\Gamma| = n^3$
- The reduction runs in time poly(n)

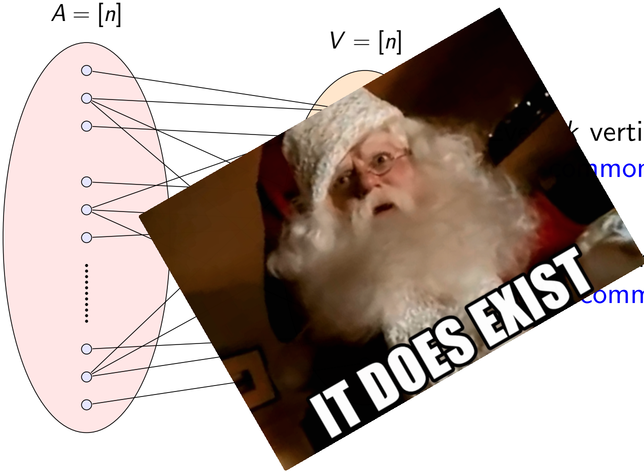
Threshold Graph



Every k vertices in V has at least $n^{1/k}$ common neighbors in A

Every $k{+}1$ vertices in V has at most $(k+1)!$ common neighbors in A

Threshold Graph



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- Erdös-Renyi model Random graphs fail: long smooth-decaying tail
- Random graphs defined over some specific ‘algebraic distribution’ suffice
- Normed graphs provide semi-explicit construction

Take-away Intuition and Remarks

- Threshold Graph Composition Technique Ingridients:
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- Threshold Graph Composition Technique Ingridients:
  - Threshold Graph
  - Composition of Input Graph with Threshold Graph
- Threshold Graph
  - What are the required threshold properties?
  - Does the graph with above properties exist?
- Tweak 'Composition of Input Graph with Threshold Graph' in order to require weaker/more realistic threshold properties
- Start from more structured Input problem

Tomorrow's plan

- Set Cover
- Biclique
- Clique